Optimization Techniques for Geometry Processing

-Part II-

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Constrained Optimization

Constrained Optimization

- general form

objective function

$f : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} \text{minimize} \quad & f(x) & & \\ \text{subject to} \quad & g_i(x) & \leq 0 & \qquad i = 1 \ldots m \end{aligned}$$

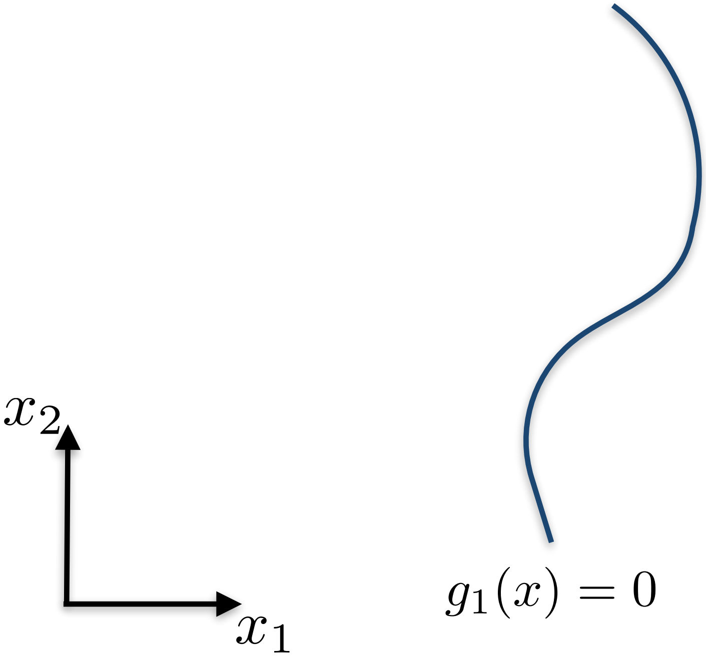
constraint functions

$g_i : \mathbb{R}^n \to \mathbb{R}$

Constrained Optimization

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & i = 1 \dots m \end{array}$$

- geometric interpretation



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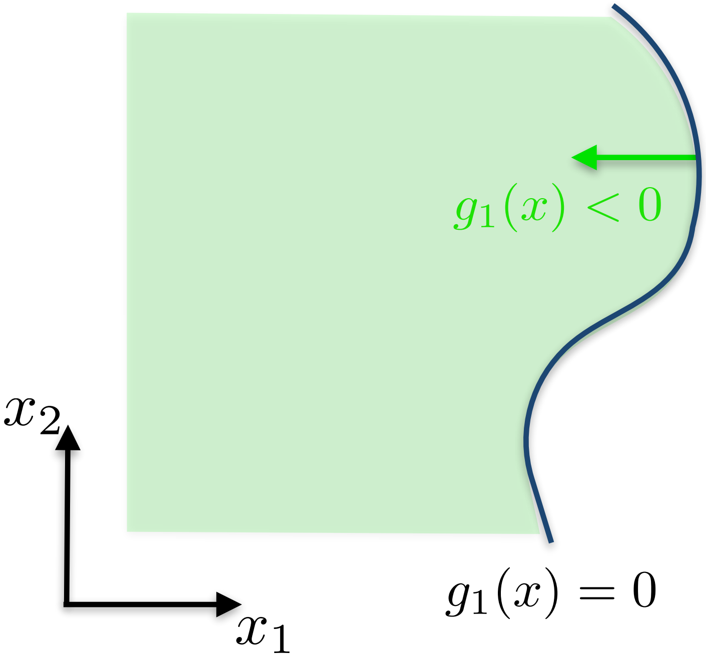

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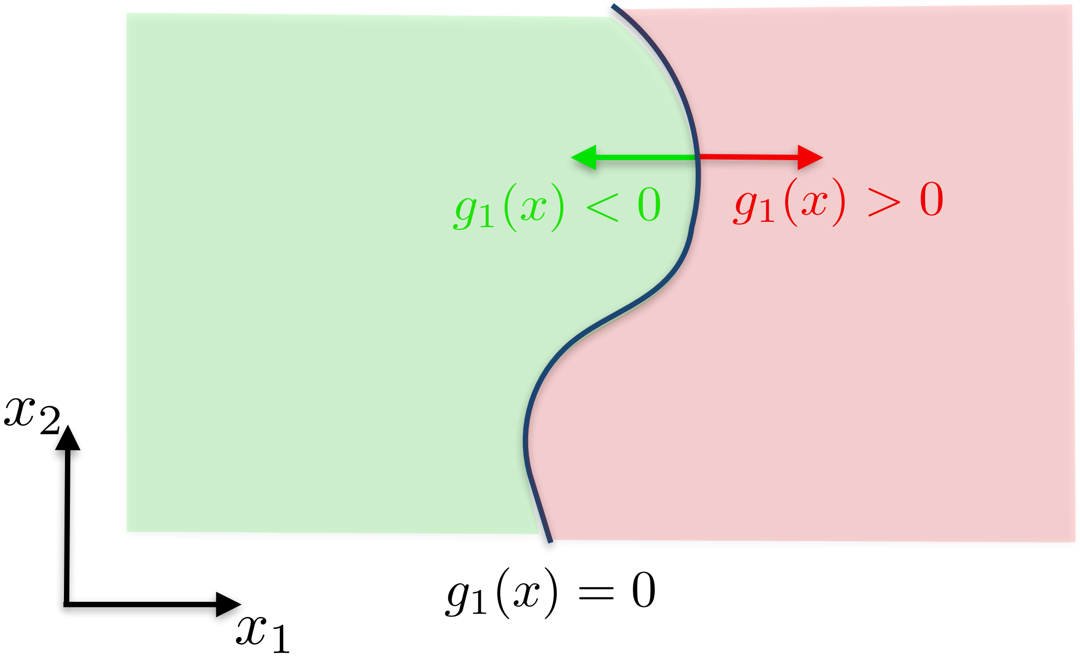

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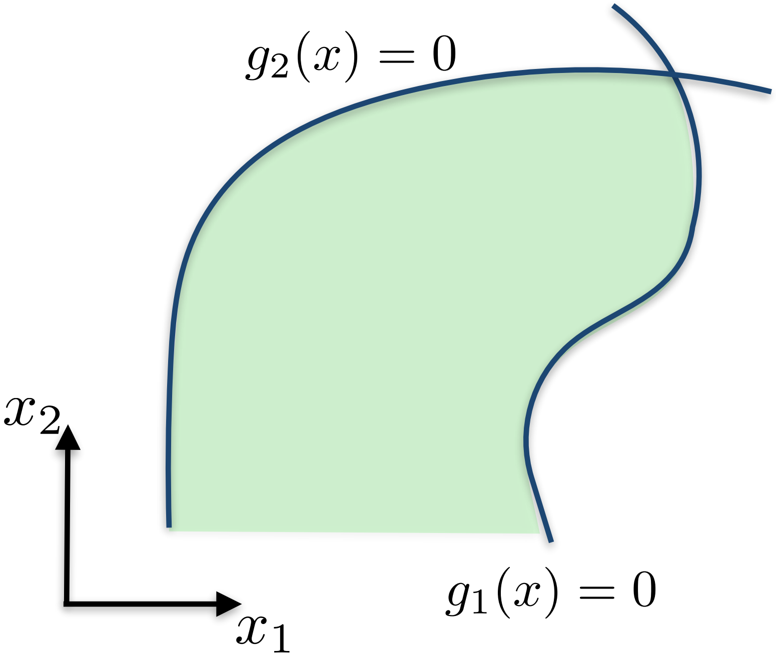

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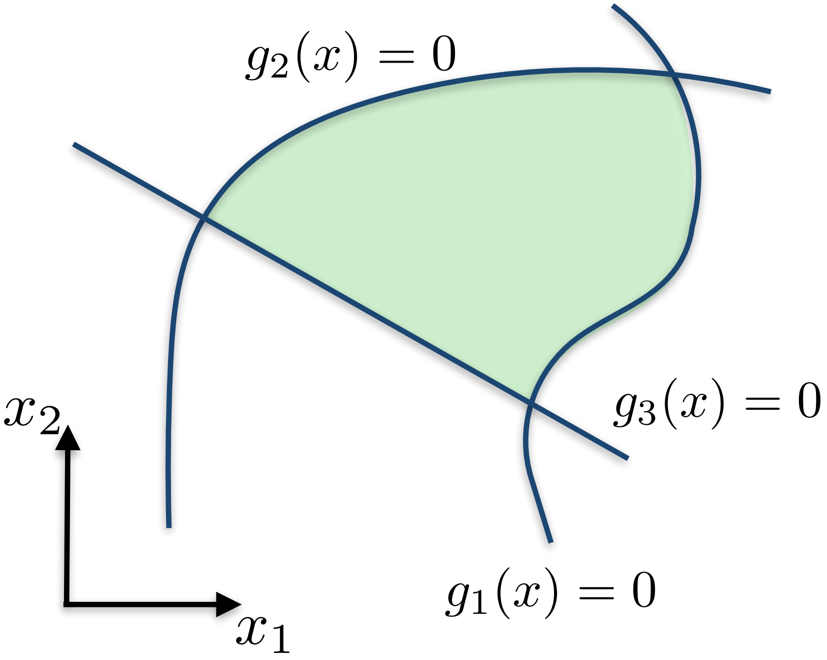

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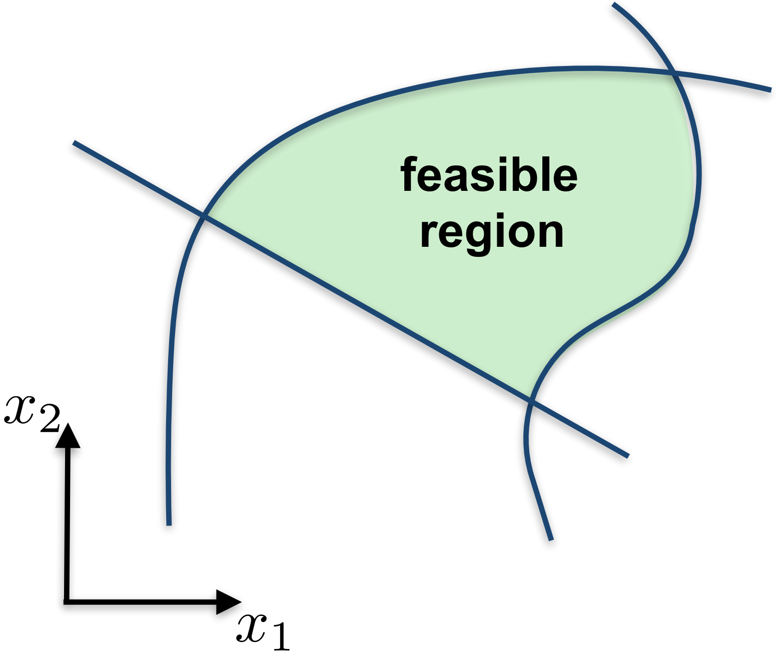

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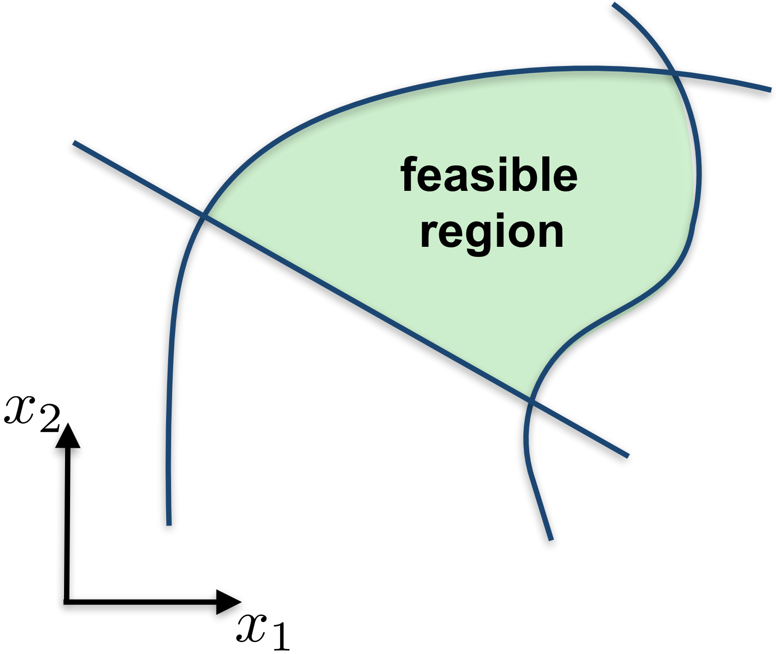

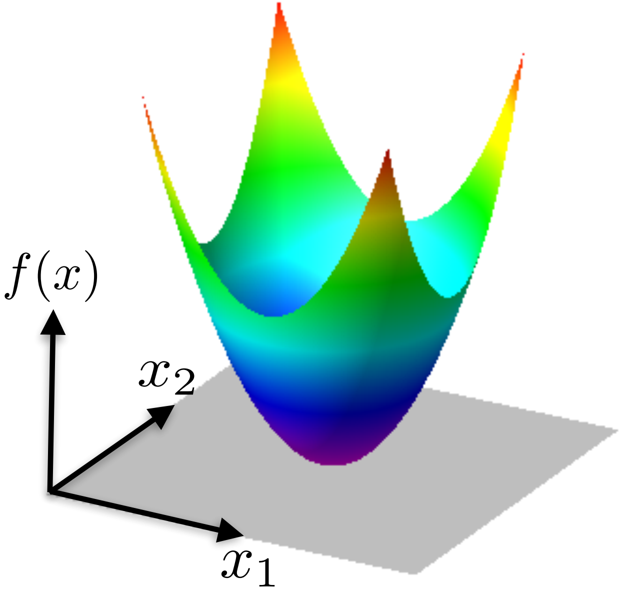

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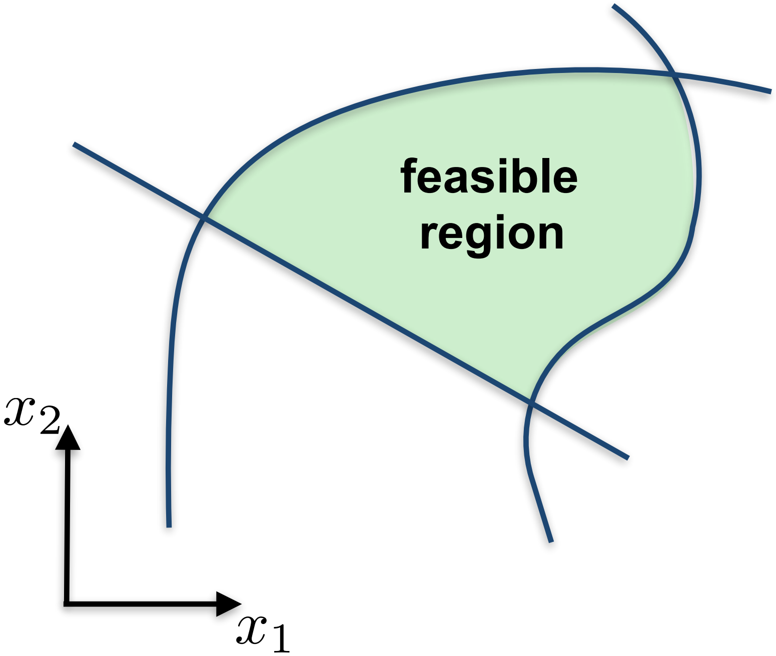

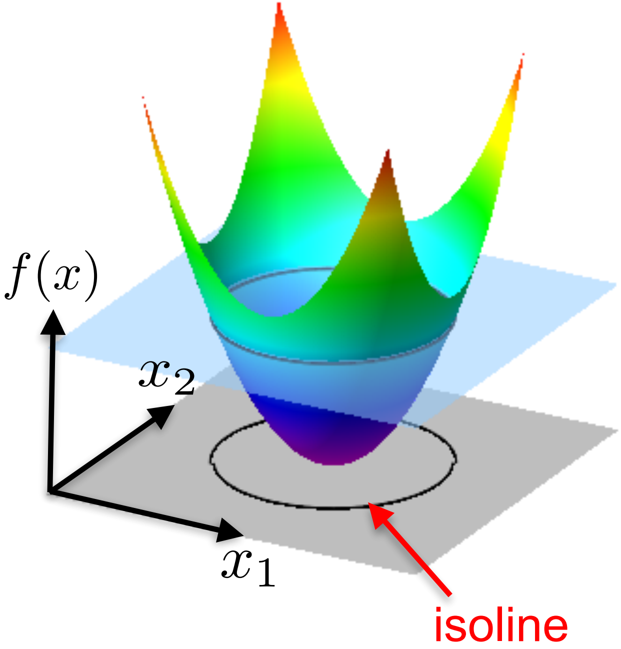

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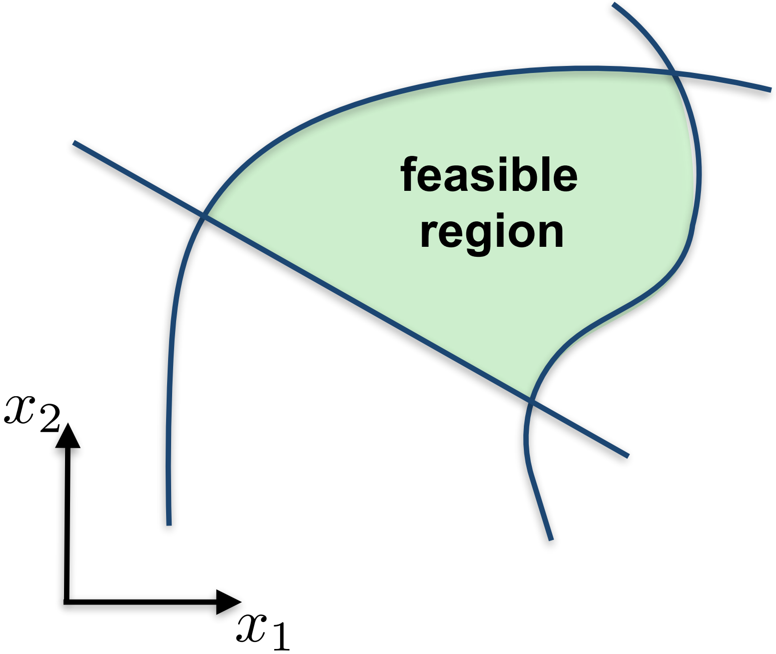

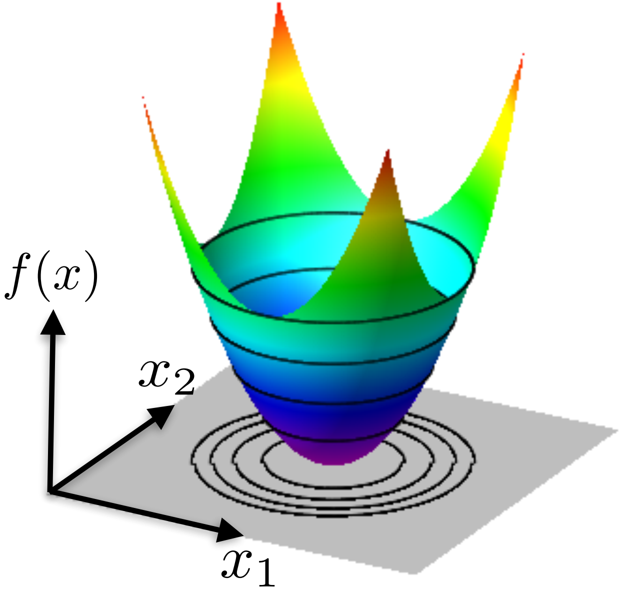

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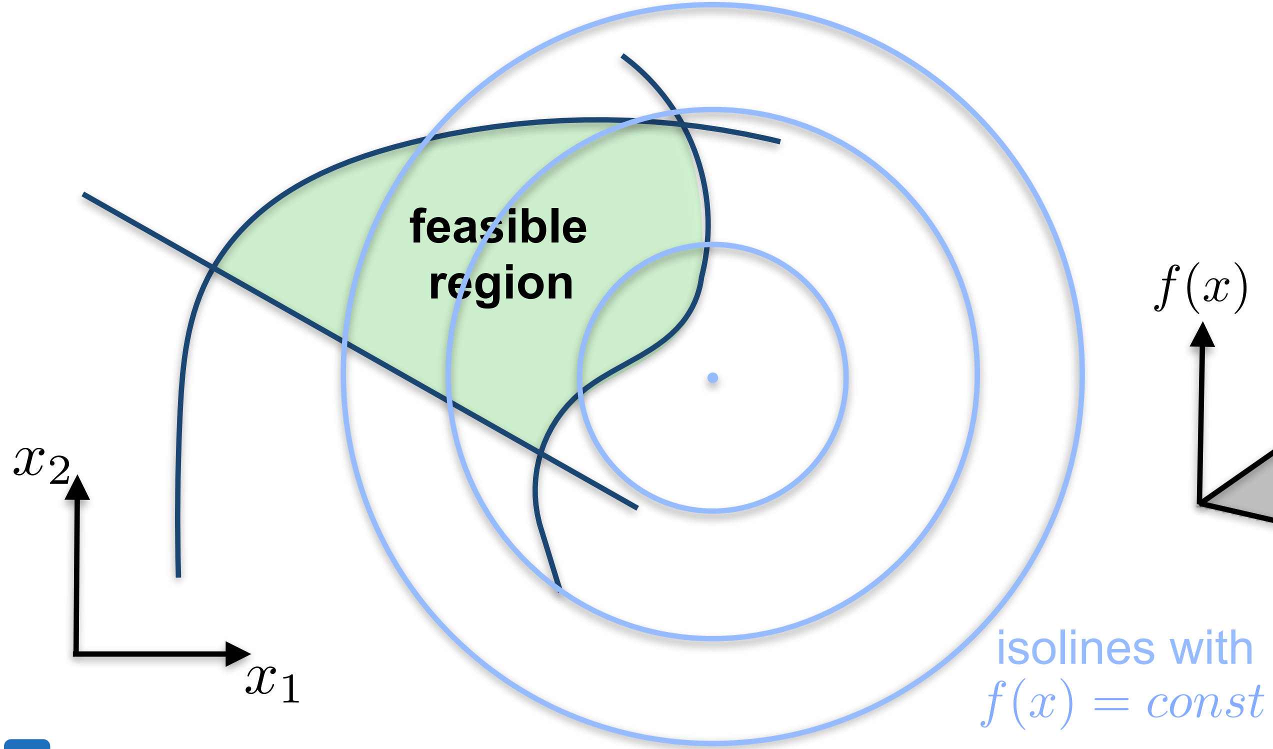

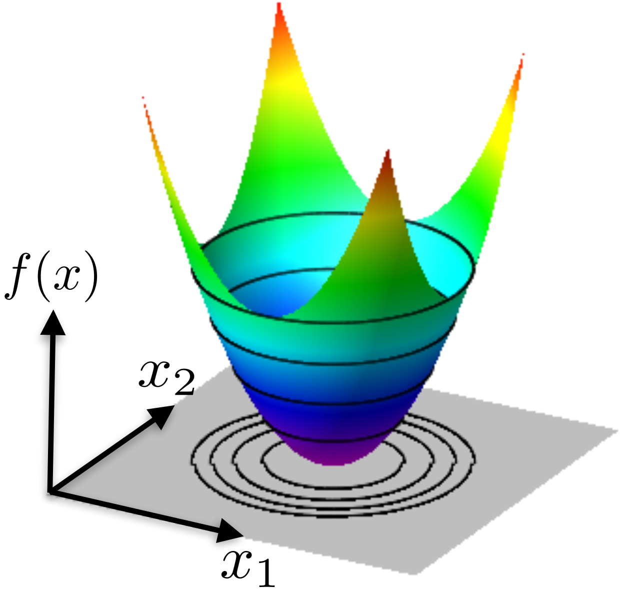

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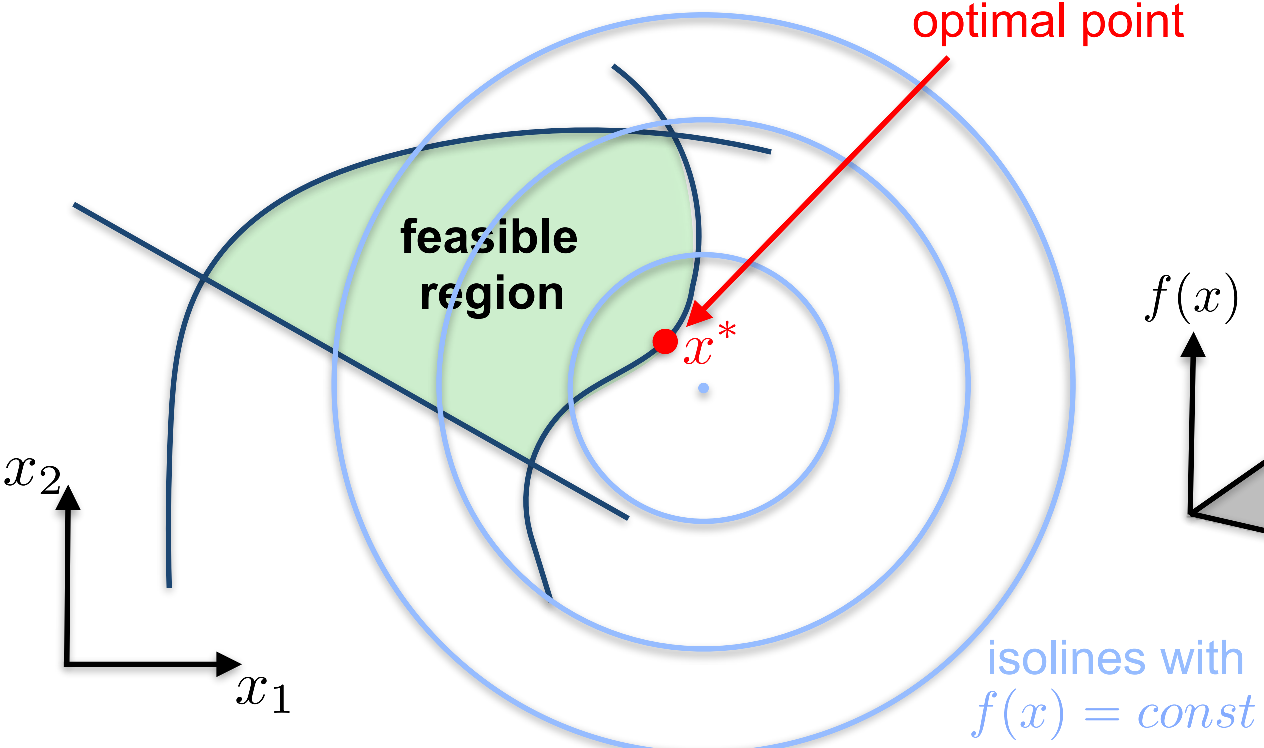
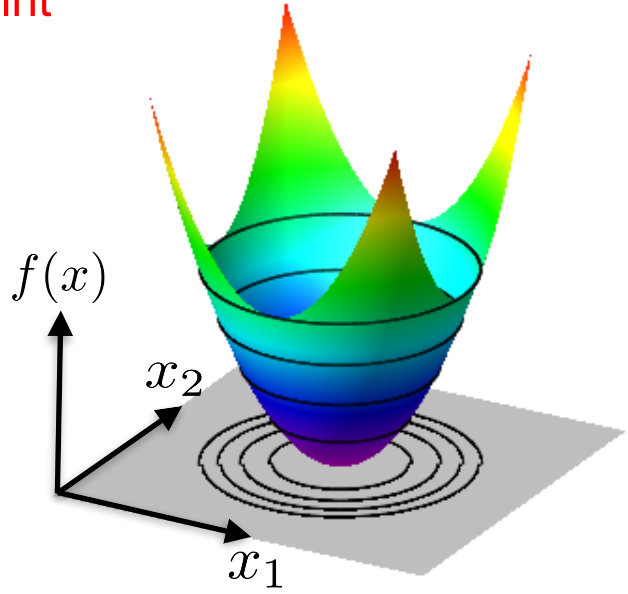
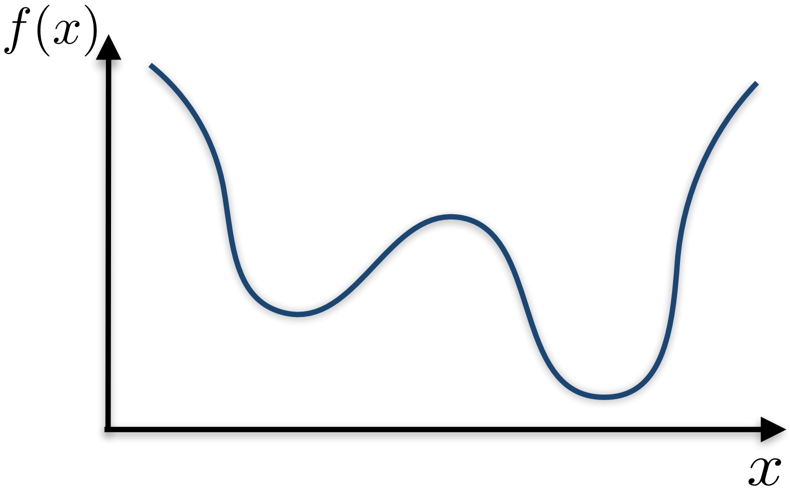




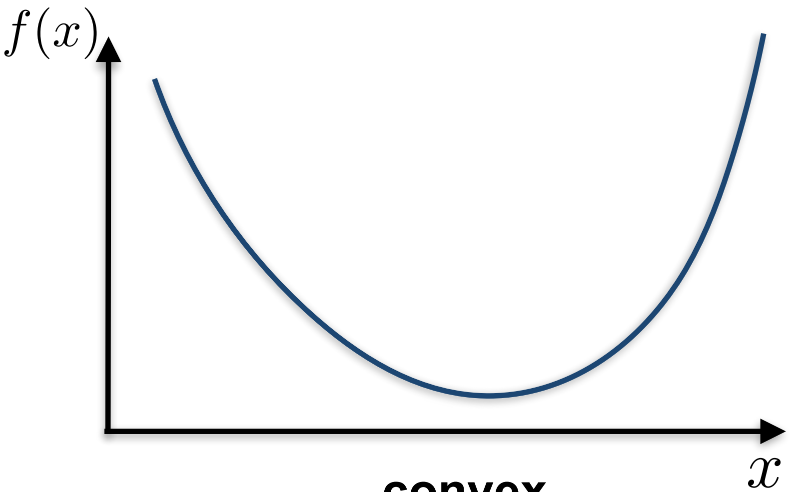
A few words on Convexity

- Searching globally optimal solutions usually requires **convexity**!



non-convex

vs.



convex

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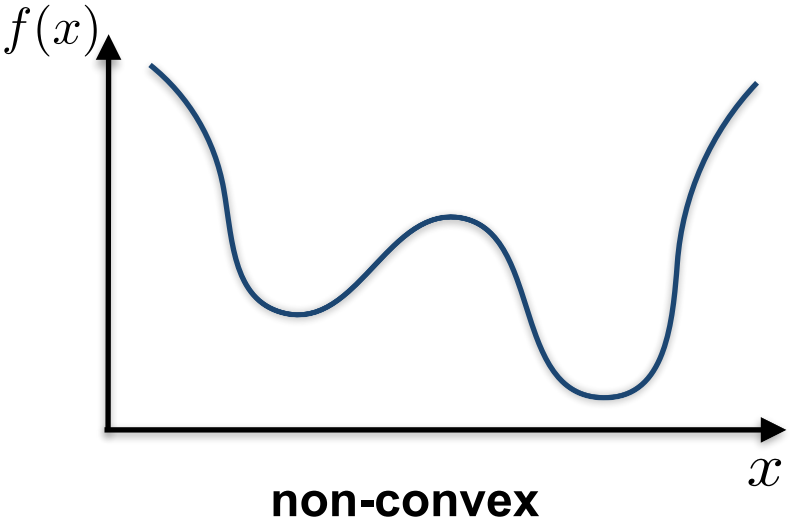

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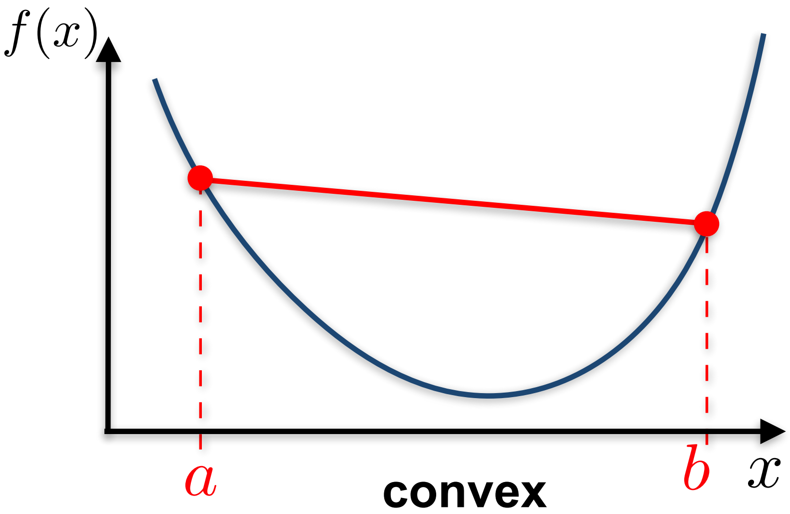

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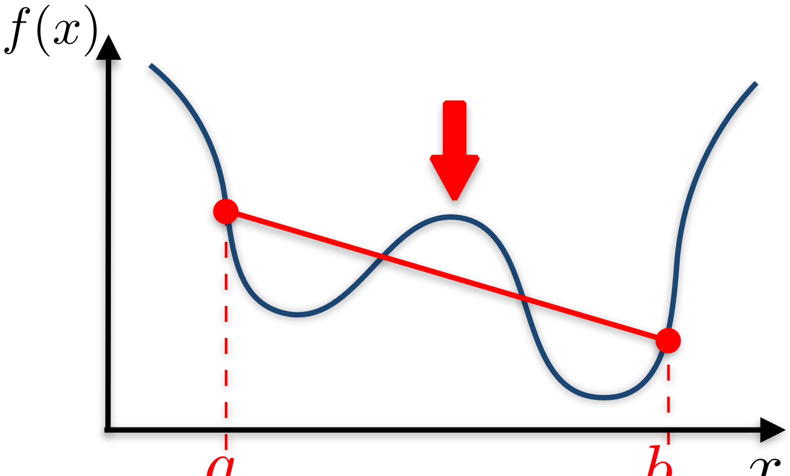

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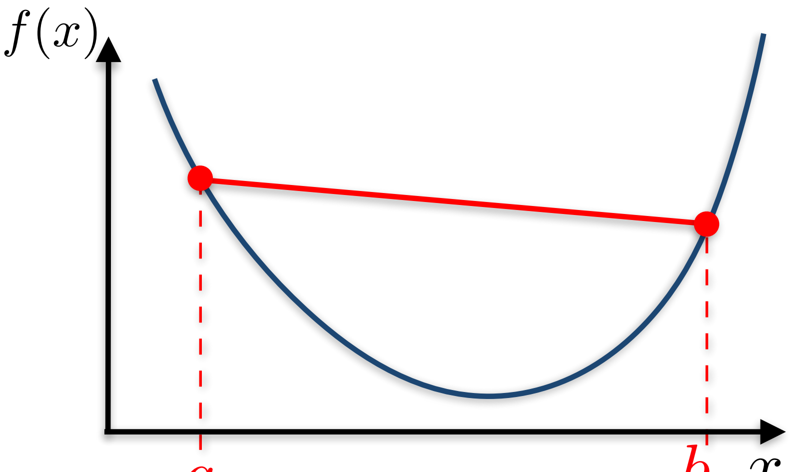

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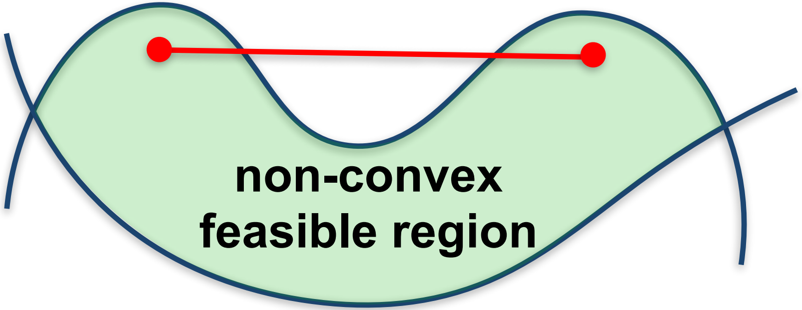
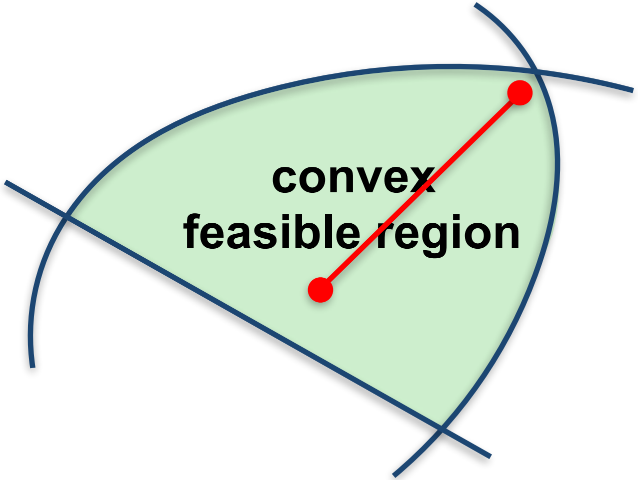
$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & i = 1 \dots m \end{array}$$

is **convex optimization problem** if $f(x)$ and all $g_i(x)$ are convex functions

consequences

- feasible region is convex set





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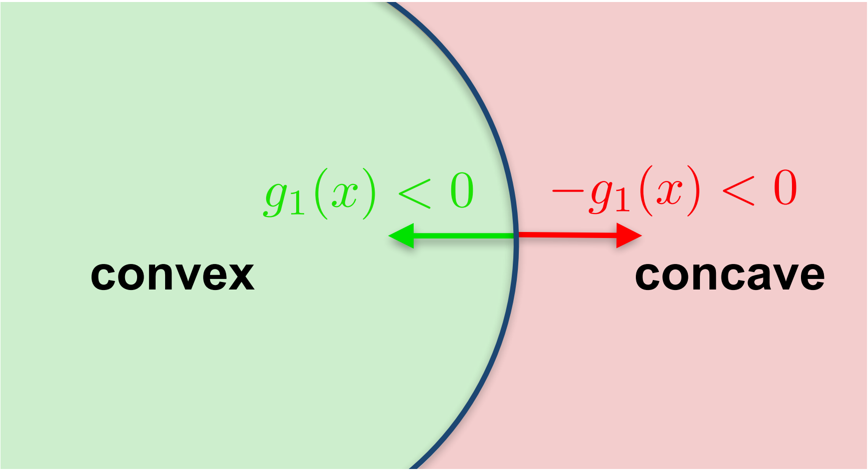

<u>consequences</u>

- feasible region is convex set
- equality constraints can only be affine, i.e. $g_i(x) = a^T x + b$ since

$$g_i(x) = 0 \iff \begin{cases} g_i(x) & \leq 0 \\ -g_i(x) & \leq 0 \end{cases}$$

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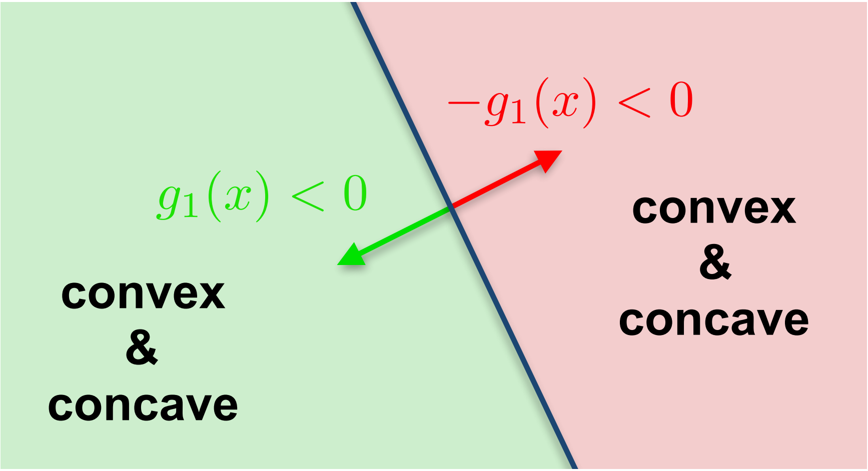


$-g_1(x) < 0$

$g_1(x) < 0$

convex
&
concave

convex
&
concave

Convexity

- Important Take Home Message:

Always Search for Convex Formulations

⇓

Global Optimum can be Found Efficiently

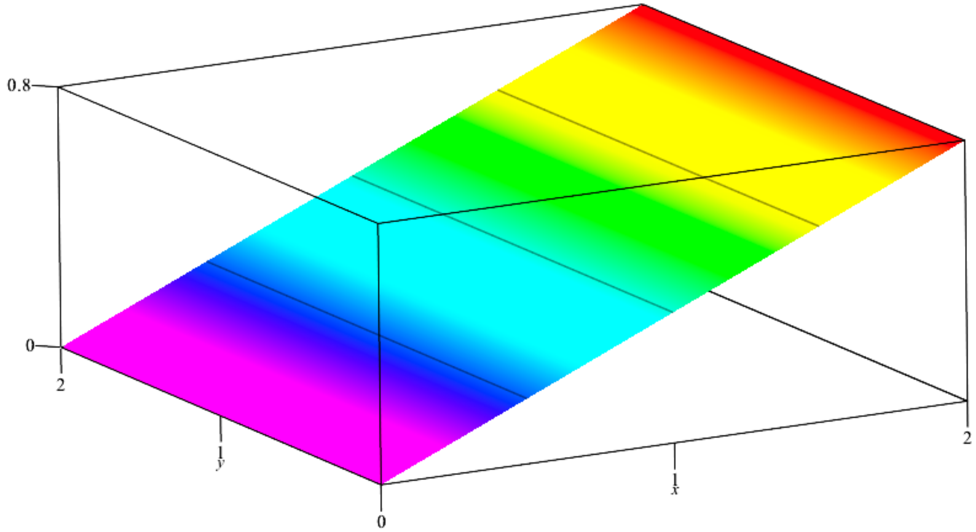
Important (convex) Problem Classes

Linear Program (LP)

- general form

linear objective

$$\begin{array}{ll} \text{minimize} & a^T x \\ \text{subject to} & Ax \leq b \end{array}$$



Linear Program (LP)

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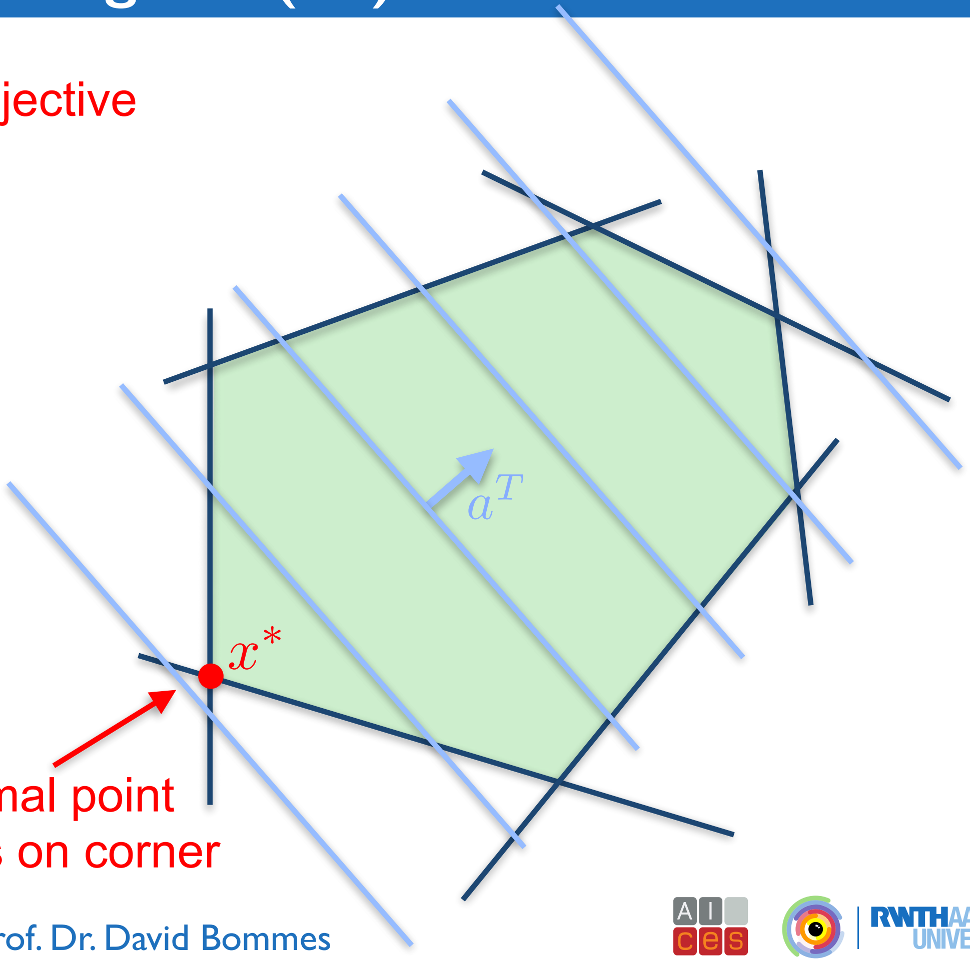
$$\begin{array}{ll} \text{minimize} & a^T x \\ \text{subject to} & Ax \leq b \end{array}$$

linear inequalities
=
intersection of halfspaces
=
polyhedron

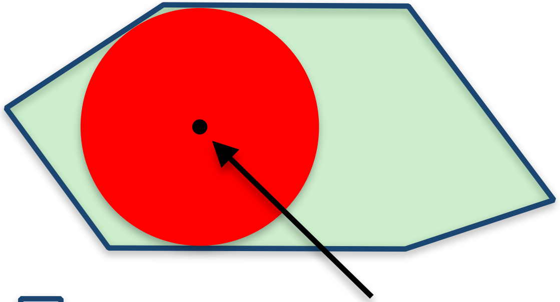


optimal point
always on corner

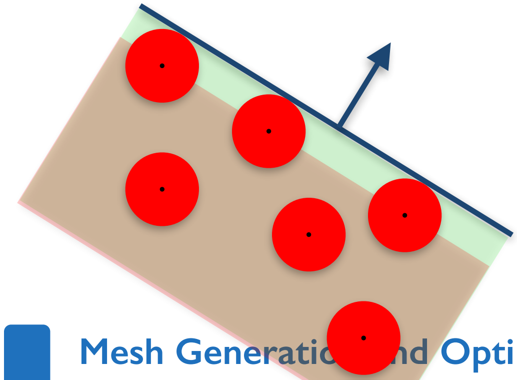
Example: Chebyshev Center

- Find largest ball inside a polyhedron in $\mathbb{R}^n$
- Ball $\mathcal{B} = \{x_c + u \mid ||u||_2 \leq r\}$
- Ball of radius r is inside halfspace

  $a^T x \leq b$ if $a^T x + \underbrace{r||a||_2}_{\text{offset}} \leq b$



Chebyshev center x_c



$$\begin{array}{lll} \text{minimize} & -r & \\ \text{subject to} & a_i^T x + r||a_i||_2 \leq b_i & i = 1 \dots m \end{array}$$

(useful) LP Reformulations

- 1-norm:

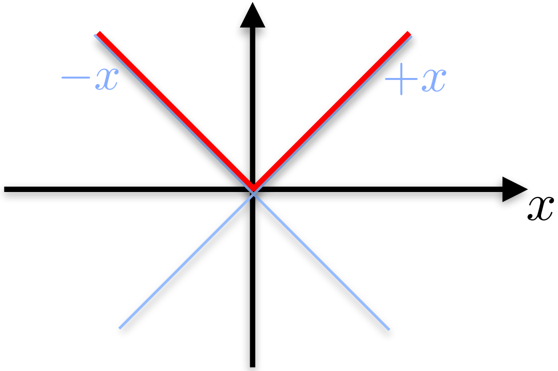
$$\text{minimize} \quad ||Ax-b||_1 = \textstyle\sum_i |a_i^T x - b_i|$$

LP

$$\begin{array}{lll} \text{minimize} & \sum_i y_i & \\ \text{subject to} & y_i \geq a_i^T x - b_i & \forall i \\ & y_i \geq -(a_i^T x - b_i) & \end{array}$$

$$y_i \geq \max \pm(a_i^T x - b_i) = |a_i^T x - b_i|$$

$$\max\{x, -x\} = |x|$$



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- ∞-norm:

$$\text{minimize} \quad ||Ax-b||_\infty = \max_i |a_i^T x - b_i|$$

LP

$$\begin{array}{llll} \text{minimize} & & y & \\ \text{subject to} & y & \geq a_i^T x - b_i & \forall i \\ & y & \geq -(a_i^T x - b_i) & \end{array}$$

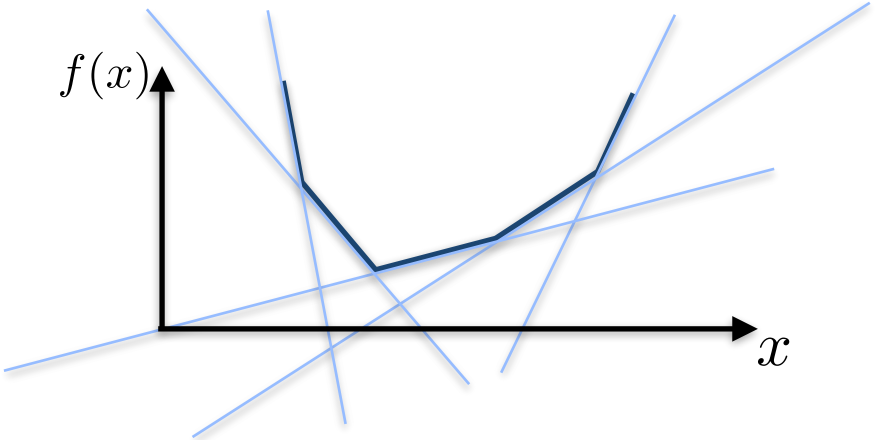
(useful) LP Reformulations

- (convex) piecewise linear function:

$$\text{minimize} \quad f(x) = \begin{cases} f_1(x) & \text{if } x \in [t_1, t_2) \\ f_2(x) & \text{if } x \in [t_2, t_3) \\ \vdots & \vdots \\ f_k(x) & \text{if } x \in [t_k, t_{k+1}) \end{cases}$$

LP

$$\text{minimize} \quad \max_i |a_i^T x - b_i|$$



- approximate arbitrarily complex convex functions by LP!
  (e.g. Outer Approximation)
- works in $\mathbb{R}^d$

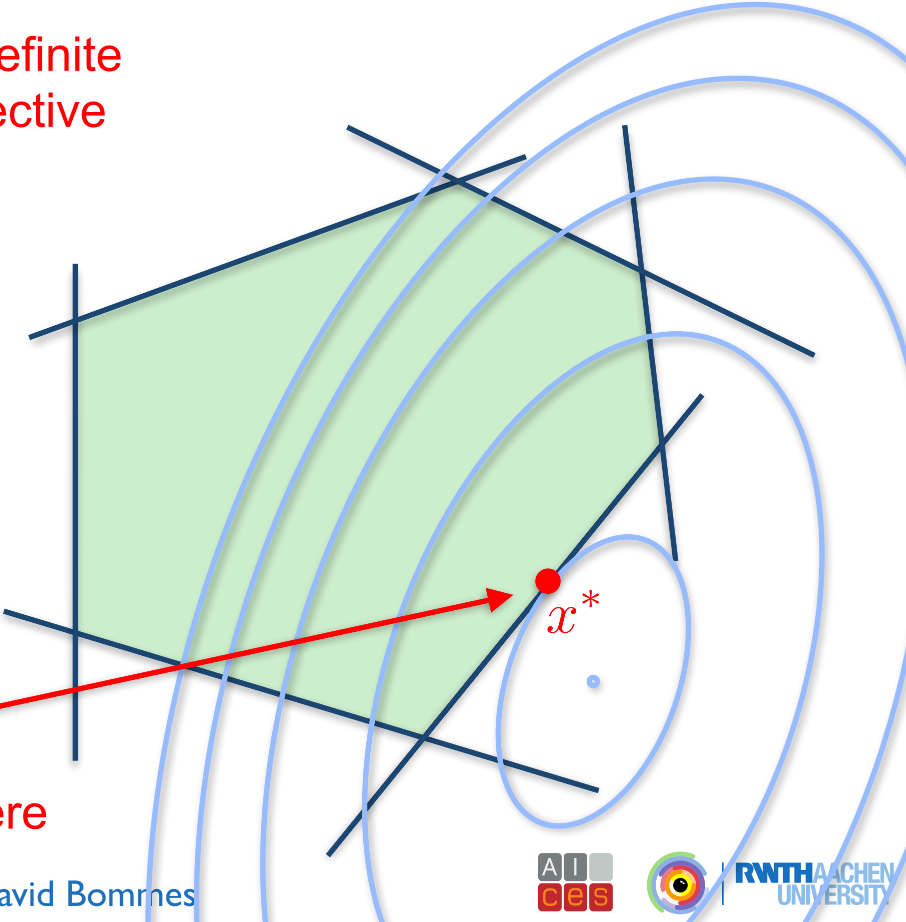
Quadratic Program (QP)

- general form

positive semidefinite quadratic objective

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}x^TQx + p^Tx + c \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

linear inequalities
=
intersection of halfspaces
=
polyhedron

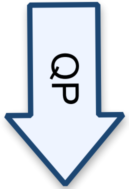


optimal point
can be anywhere

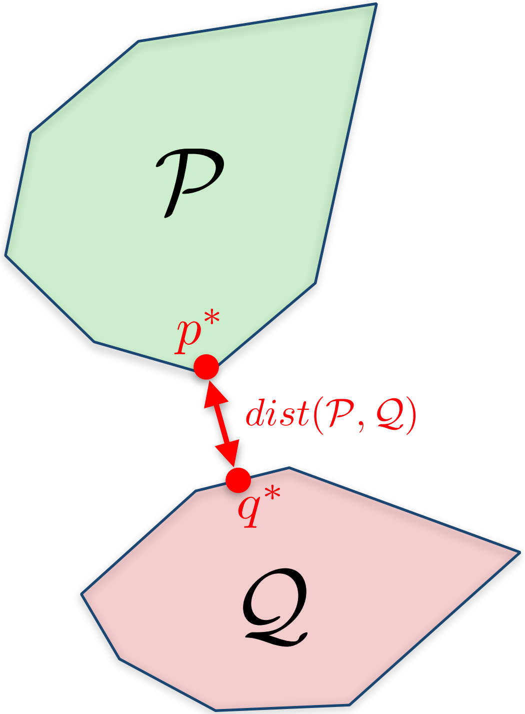
Example QP

- Distance between polyhedra:

$$dist(\mathcal{P},\mathcal{Q}) = \inf\{||p-q||_2 \mid p\in\mathcal{P},\ q\in\mathcal{Q}\}$$



$$\begin{array}{lll} \text{minimize} & ||p-q||_2^2 & \text{with } x=(p,q) \\ \text{subject to} & A_P p & \leq b_P \\ & A_Q q & \leq b_Q \end{array}$$



Second-Order Cone Program (SOCP)

- general form

$$\begin{array}{lll} \text{minimize} & a^T x & \\ \text{subject to} & ||A_i x + b_i||_2 \leq c_i^T x + d_i & i = 1 \ldots m \end{array}$$

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- $\mathrm{LP} \subset \mathrm{QP} \subset \mathrm{QCQP} \subset \mathrm{SOCP}$

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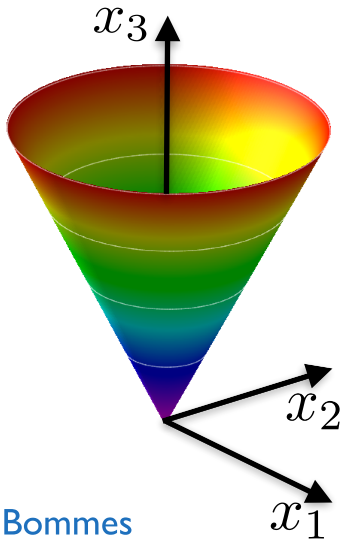
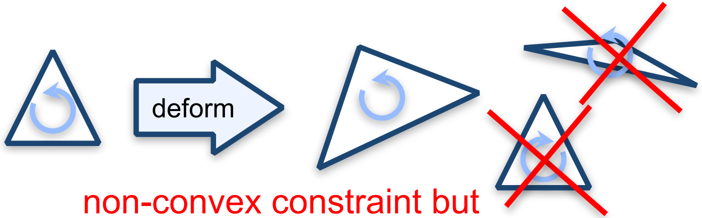
- example constraint:

$$\sqrt{x_1^2 + x_2^2} \leq x_3$$

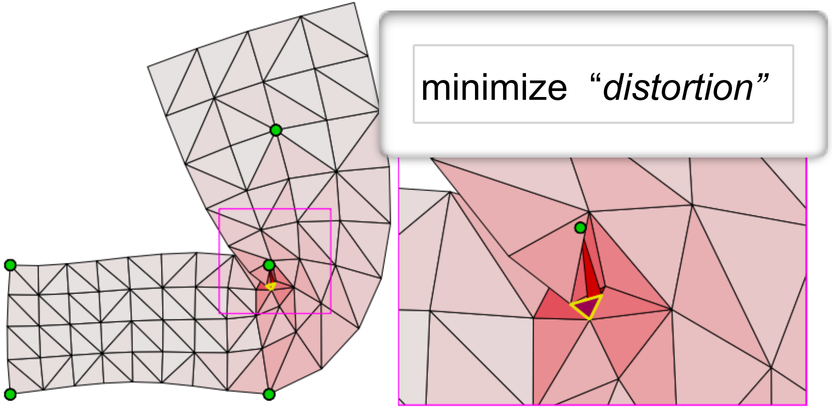


Example SOCP

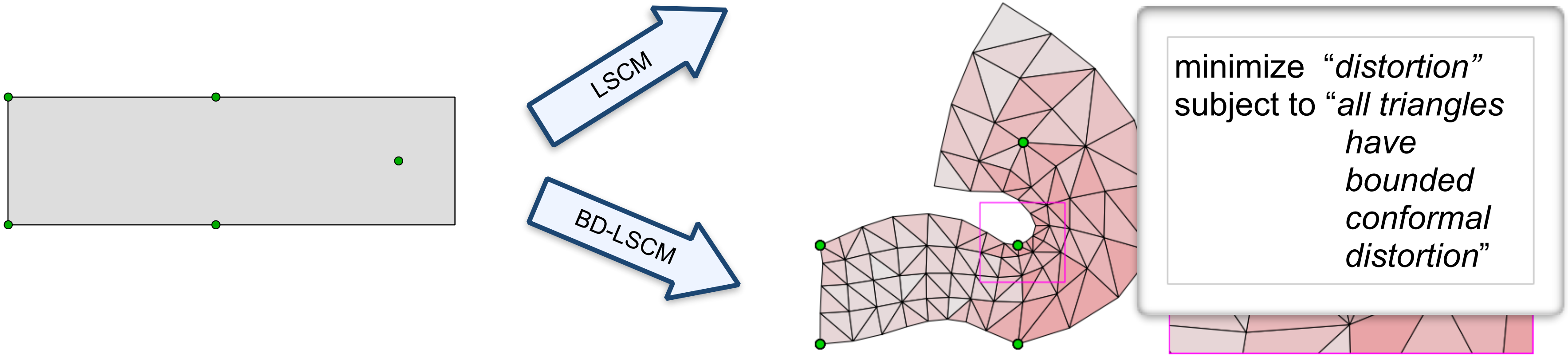
- Bounded (conformal) distortion mappings in 2D **[Lipman 2012]**:



non-convex constraint but approximation by SOCP possible!



minimize *“distortion”*



minimize *“distortion”*
subject to *“all triangles have bounded conformal distortion”*

Semidefinite Program (SDP)

- general form

$$\begin{array}{lll} \text{minimize} & \operatorname{tr}(CX) & = \sum_{i,j} C_{ij} X_{ij} \\ \text{subject to} & \operatorname{tr}(A_i X) & \leq b_i \qquad i = 1 \ldots m \\ & X & \succeq 0 \end{array}$$

linear function in matrix space

with $X, C, A_i \in S^n$
(symmetric nxn matrices)

X is required to be **positive semidefinite**

Finding Minima

First-Order Optimality Conditions

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & i = 1 \dots m \end{array}$$

- **Necessary condition for minimum of**

- **Langrangian:** $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$

- **Karush-Kuhn-Tucker (KKT)** conditions for **minimum** x^*
  1. Stationarity: $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0$
  2. Primal feasibility: $g_i(x^*) \leq 0$
  3. Dual feasibility: $\lambda_i \geq 0$
  4. Complementary slackness: $\lambda_i g_i(x^*) = 0$

without constraints just

$$\nabla f(x) = 0$$

First-Order Optimality Conditions

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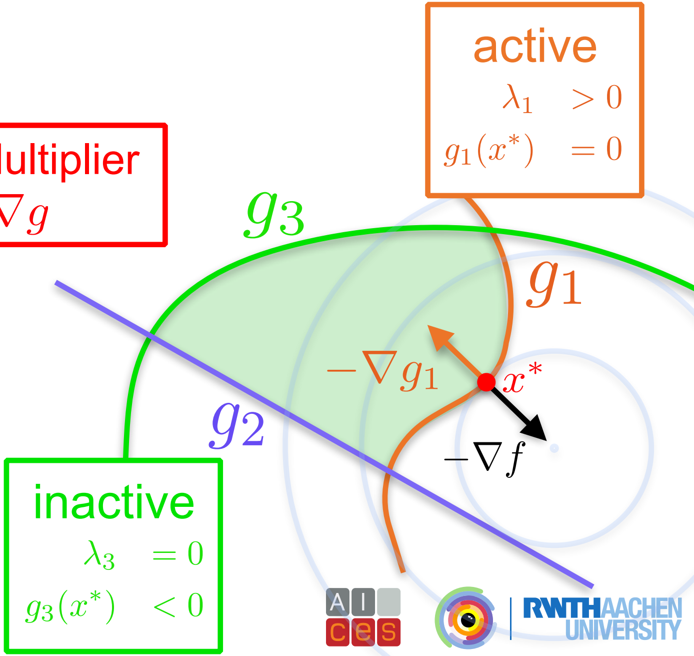
Recall Lagrange Multiplier
$\nabla f = -\lambda \nabla g$



active
$\lambda_1 > 0$
$g_1(x^*) = 0$

inactive
$\lambda_3 = 0$
$g_3(x^*) < 0$

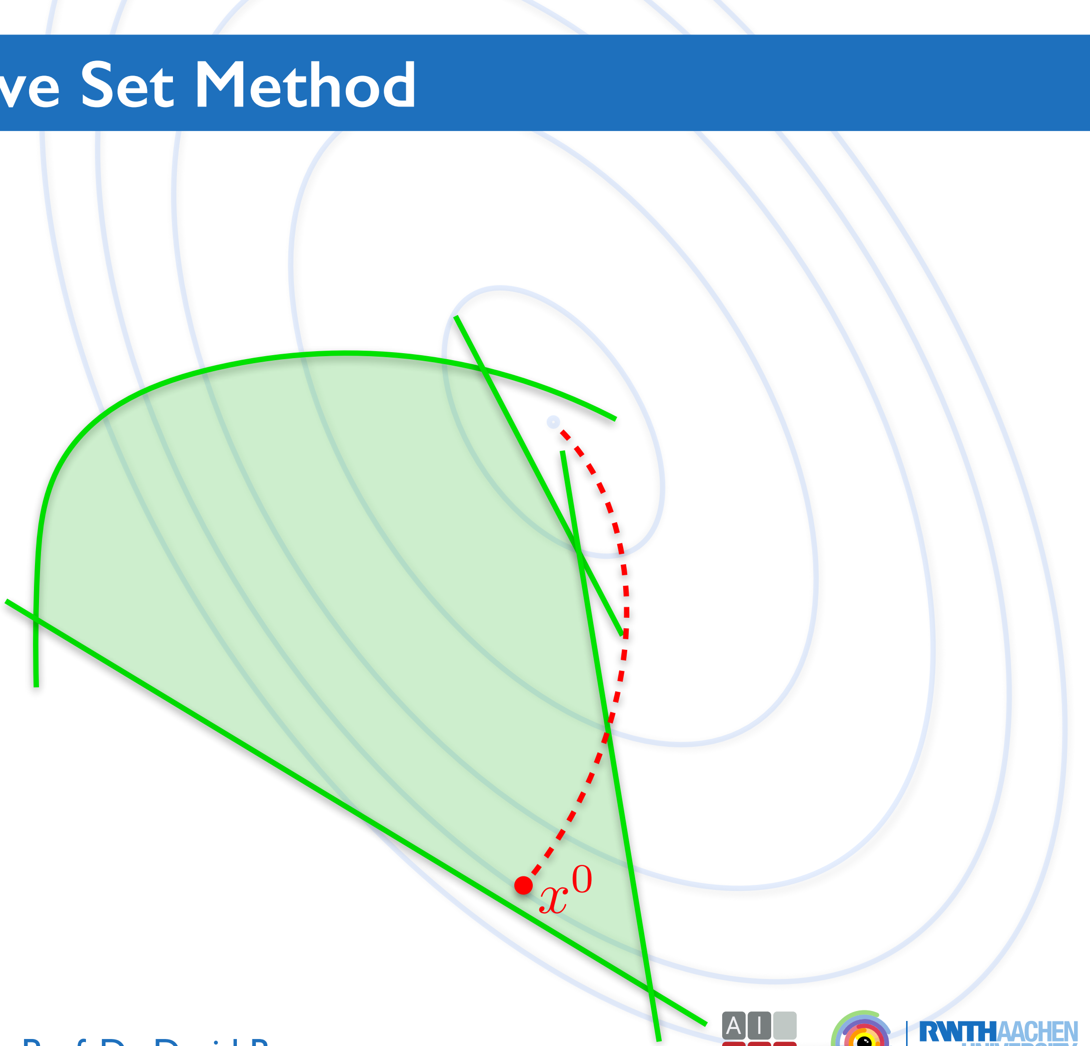
Algorithms

Active Set Method

Active Set Method

- Assume feasible x^0
- Initialize Active Set $A = \{\}$

- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it



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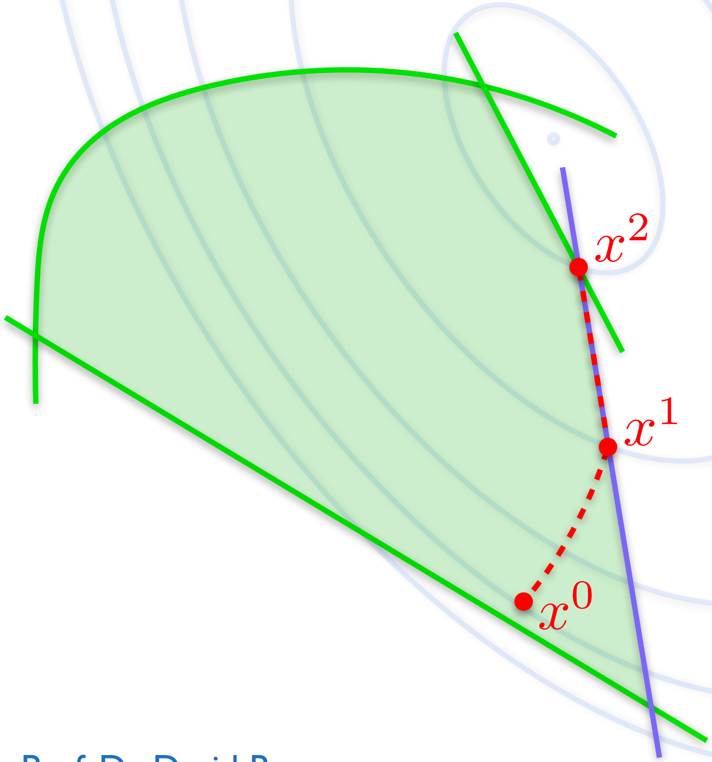

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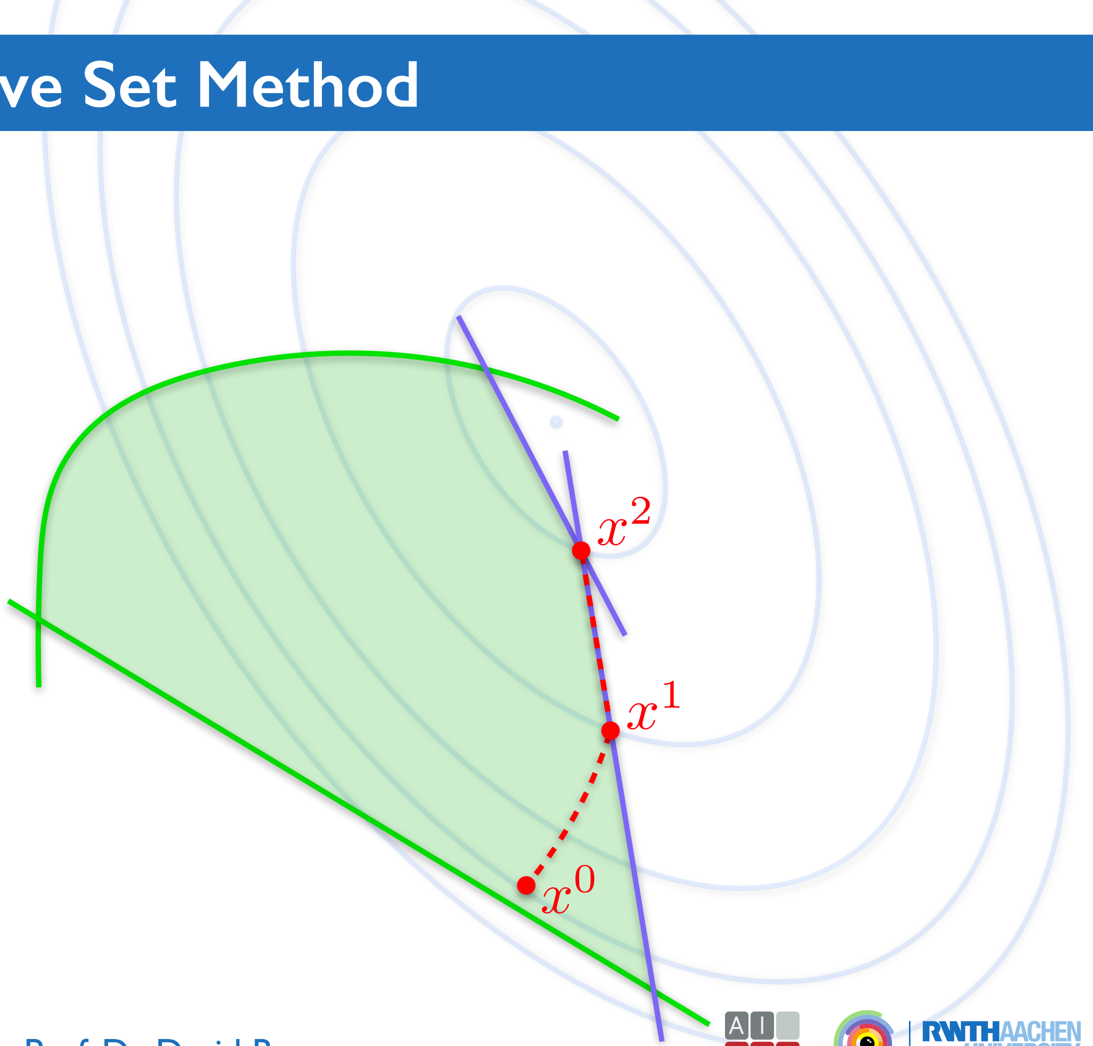

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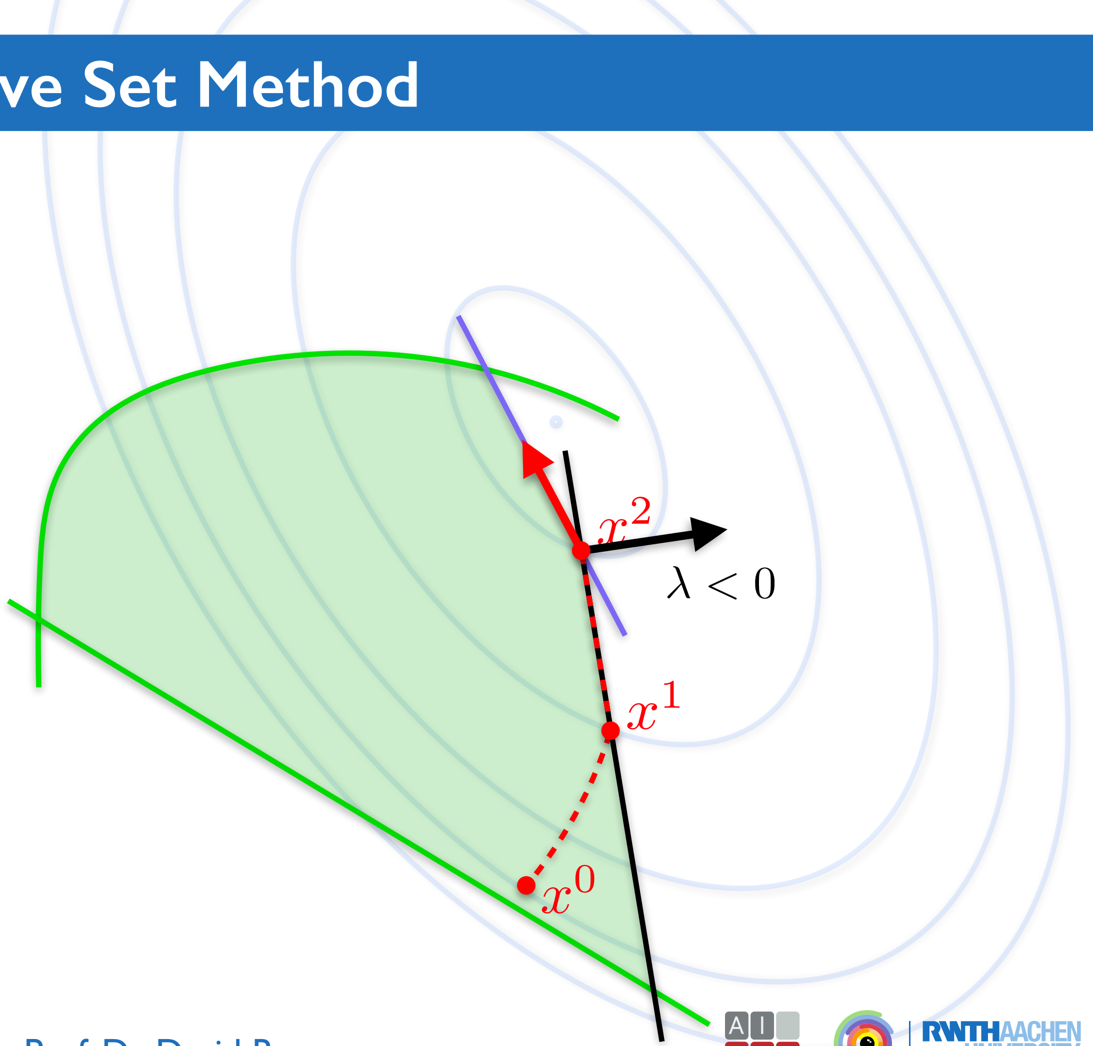

Active Set Method

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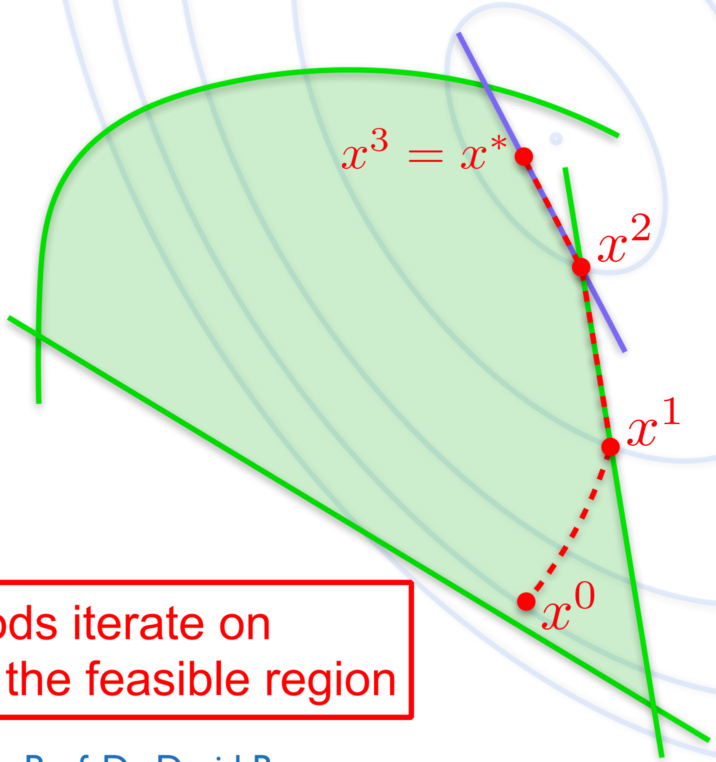
- Initialize Active Set $A = \{\}$

- While KKT not satisfied
  - Optimize with equality constraints of A
  - if constraint gets violated, **activate** it
  - if Lagrange multiplier gets negative **deactivate** corresponding constraint

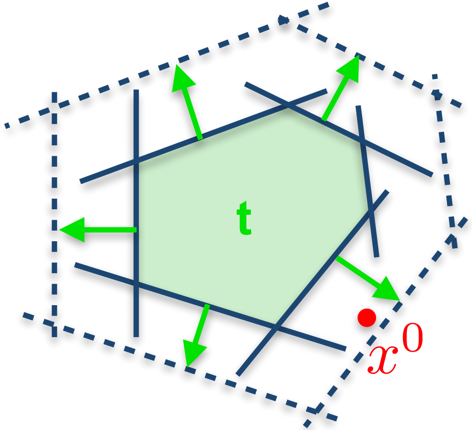


active set methods iterate on the boundary of the feasible region

How to find feasible starting point?

- Given

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & i = 1 \ldots m \end{array}$$



- Corresponding (Phase 1) **Feasibility Problem**:

$$\begin{array}{lll} \text{minimize} & t & \\ \text{subject to} & g_i(x) \leq t & i = 1 \ldots m \end{array}$$

also constrained problem **but** for arbitrary $x^0 \in \mathbb{R}^n$ simply find feasible starting point via

$$t^0 = \max_i g_i(x^0)$$

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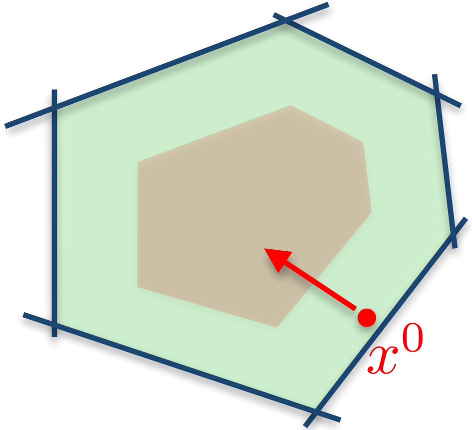

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$$t^0 = \max_i g_i(x^0)$$

terminate as soon as t < 0 !!!

Active Set Method

- Many (sophisticated) variants
  - primal, dual, primal-dual, …
- Many specializations
  - LP (e.g. simplex algorithm), QP, NLP, …
- Good choice if
  - few constraints
  - warmstart desired
- Disadvantage
  - sometimes requires huge number of iterations

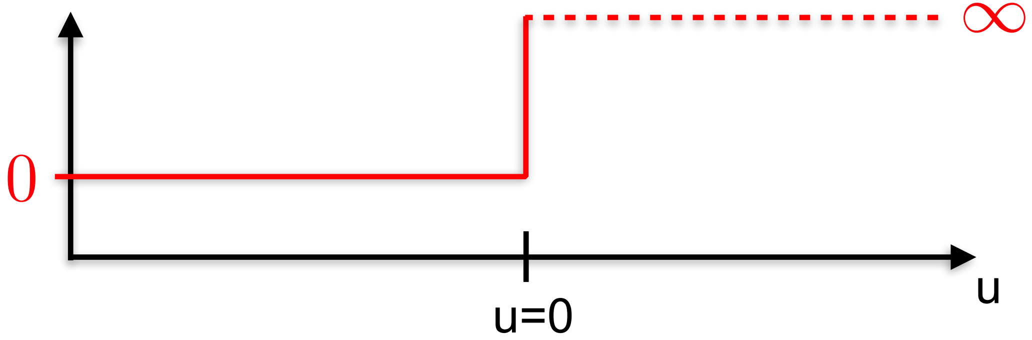
Interior Point Method

Interior Point Method

- Sometimes called **Barrier Method**

- **Barrier function**

$$I_{-}(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \infty & \text{if } u > 0 \end{cases}$$



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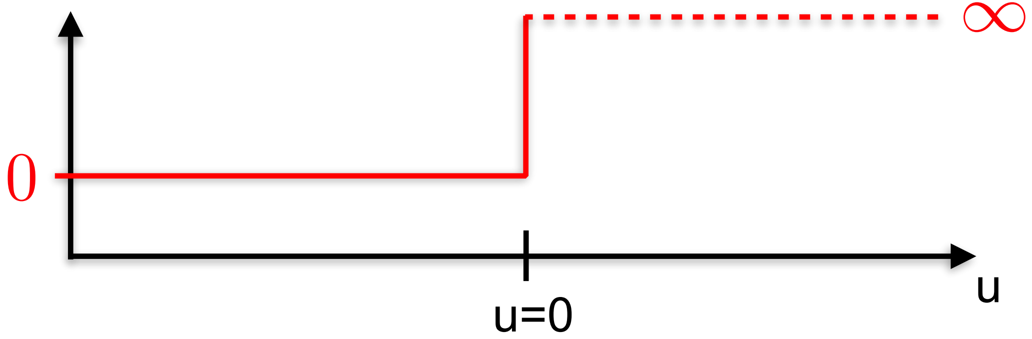
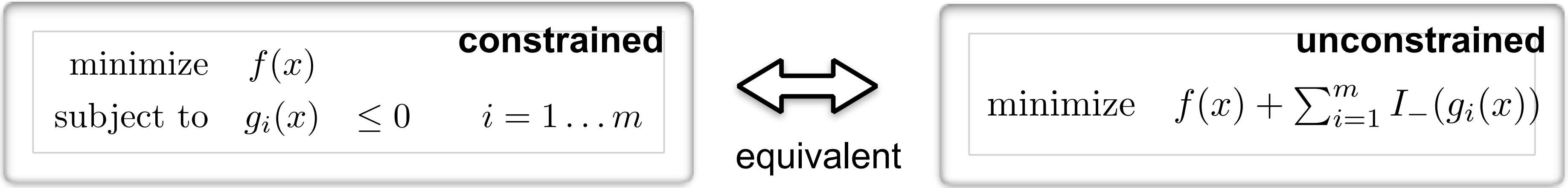


- **Idea:**



constrained

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0 & i = 1 \ldots m \end{array}$$

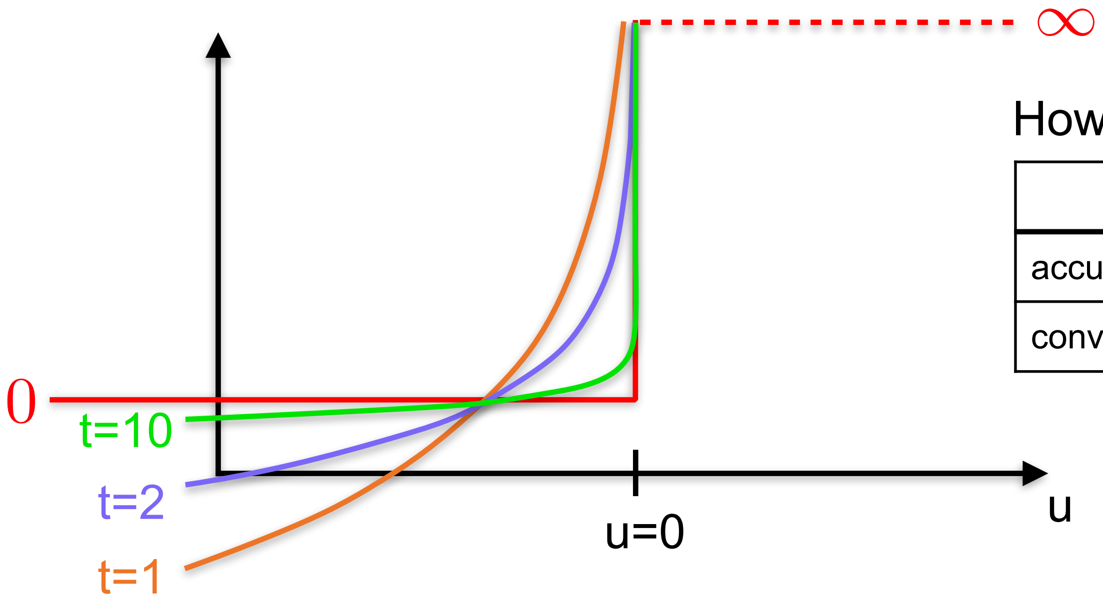
unconstrained

$$\text{minimize} \quad f(x) + \sum_{i=1}^{m} I_-(g_i(x))$$

Barrier Function

- **Logarithmic Barrier** (smooth and convex approximation)

$$\hat{I}_{-}(u) = -\frac{1}{t}\log(-u)$$



How to choose t?

	large t	small t
accuracy	+	-
convergence	-	+

Simple Interior Point Method

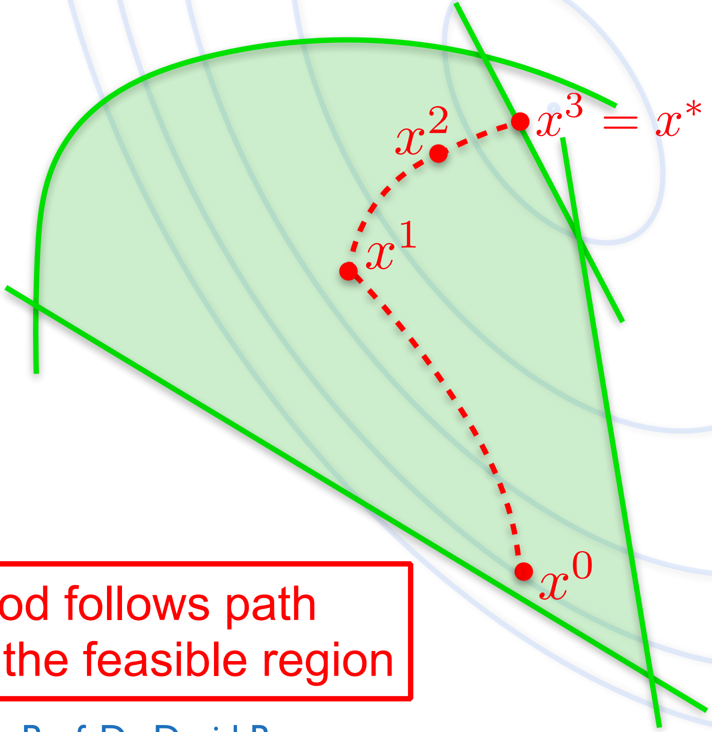
- Assume feasible x^0 , $t := 1$

- **Iteratively solve** (unconstrained)

$$\text{minimize} \quad f(x) - \frac{1}{t}\sum_{i=1}^{m} \log(-g_i(x))$$

and update $t := 10 \cdot t$

- stop when $\frac{m}{t} < \epsilon$



interior point method follows path through interior of the feasible region

Interior Point Method

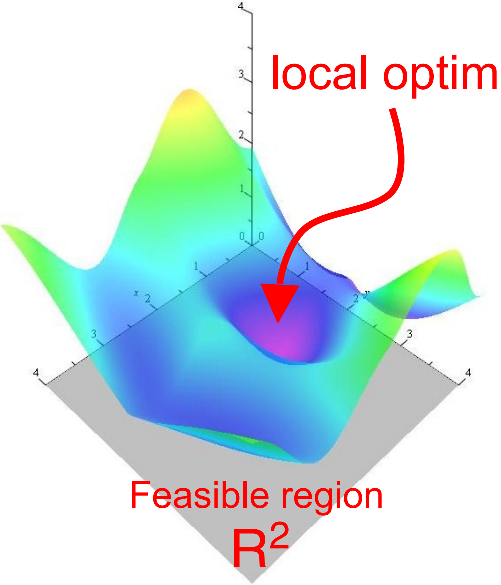
- Many (sophisticated) variants
  - primal, dual, primal-dual, …
  - adaptive choice of t
- Many specializations
  - LP, QP, NLP, …
- Good choice if
  - large scale (sparse) problem with many constraints
  - non-convex feasible region
- Disadvantage
  - limited warmstart capabilities

More Algorithms …

Mixed-Integer Optimization

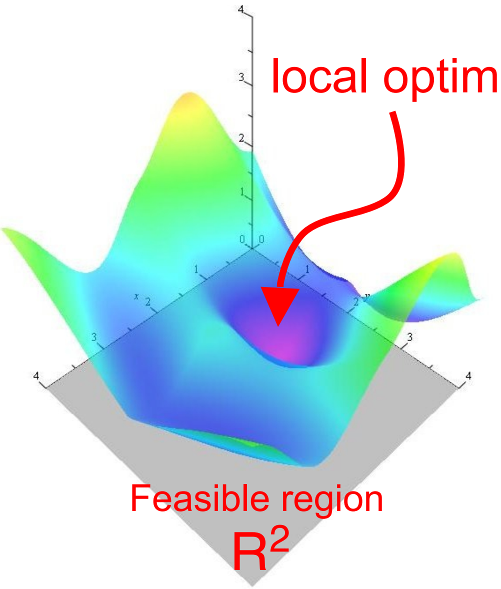
Mixed-Integer Optimization

- Minimize objective $f(\underbrace{x_1 \dots x_n}_{\in \mathbb{R}^n})$



Mixed-Integer Optimization

- Minimize objective $f(\underbrace{x_1 \dots x_n}_{\in \mathbb{R}^n}, \underbrace{y_1 \dots y_d}_{\in \mathbb{Z}^d})$



local optimum

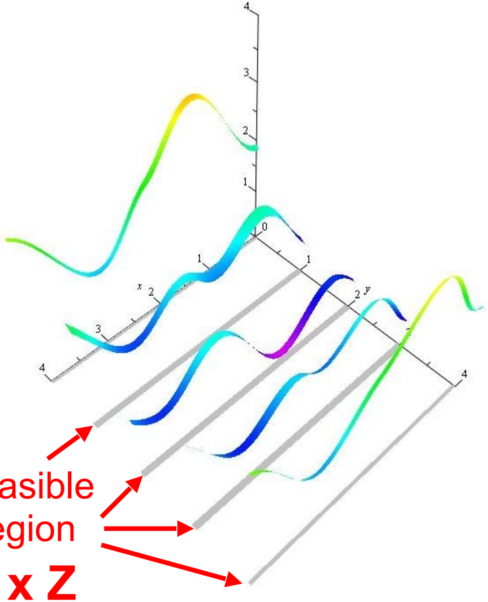
Feasible region

R^2

continuous

vs.

mixed-integer



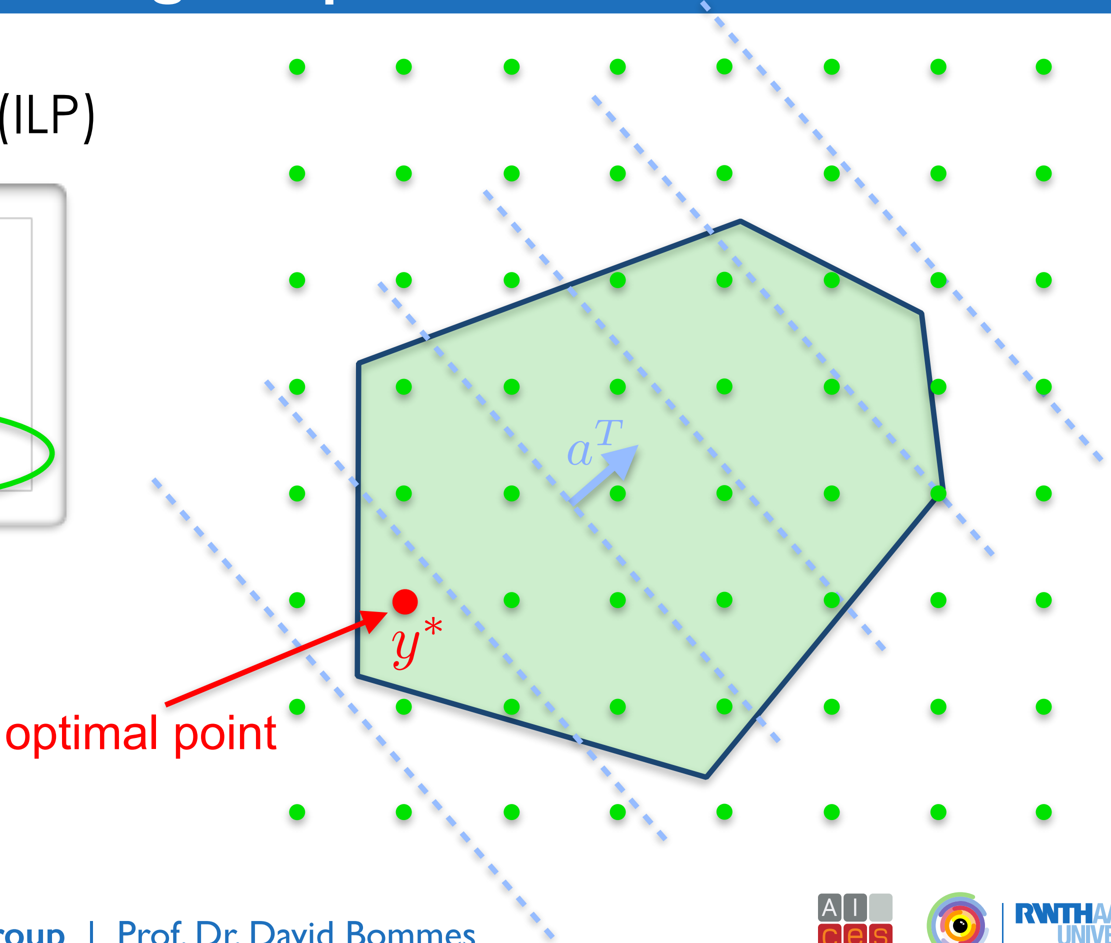
Feasible region

R x Z

Mixed-Integer Optimization

- Integer Linear Program (ILP)

$$\begin{array}{rl} \text{minimize} & a^T y \\ \text{subject to} & Ay \leq b \\ & y \in \mathbb{Z}^d \end{array}$$



Branch and Bound

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- **Bounding**: initially we only know $f(x^*, y^*) \in (-\infty, \infty)$

lower bound upper bound

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- **Idea:** improve bounds by series of **continuous problems**
- **Continuous Relaxation:** assume that all variables are continuous

$$\boxed{\begin{array}{lll} \text{minimize} & f(x,y) & \textbf{MIP} \\ \text{subject to} & g_i(x,y) & \leq 0 \\ & y \in \mathbb{Z}^d & \end{array}} \Rightarrow \boxed{\begin{array}{lll} \text{minimize} & f(x,y) & \textbf{RMIP} \\ \text{subject to} & g_i(x,y) & \leq 0 \\ & \xcancel{y \in \mathbb{Z}^d} & \end{array}}$$

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MIP

$$\begin{array}{lll} \text{minimize} & f(x,y) & \\ \text{subject to} & g_i(x,y) & \leq 0 \\ & y \in \mathbb{Z}^d & \end{array}$$

$$\Rightarrow$$

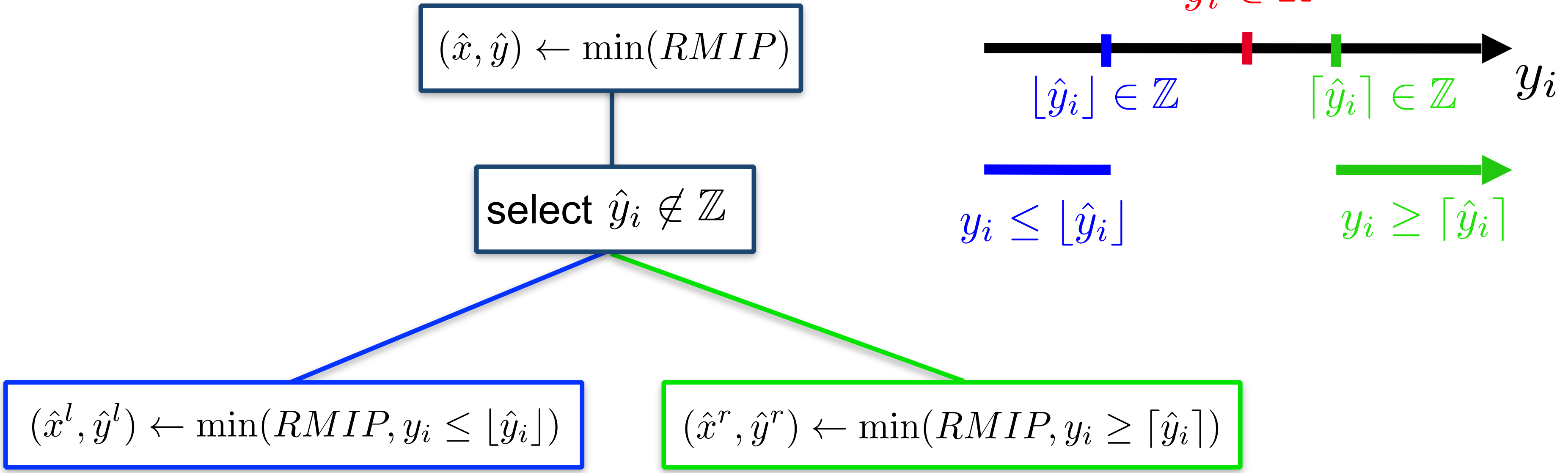
RMIP

$$\begin{array}{lll} \text{minimize} & f(x,y) & \\ \text{subject to} & g_i(x,y) & \leq 0 \\ & \xcancel{y \in \mathbb{Z}^d} & \end{array}$$

- MIP is restriction of RMIP $\Rightarrow$ lower bound $\min(MIP) \geq \min(RMIP)$
- How to obtain upper bound? How to improve bounds?

Branch and Bound

- **Branching:**



cut off non-integer neighborhood of $\hat{y}_i \in \mathbb{R}$

$(\hat{x},\hat{y}) \leftarrow \min(RMIP)$

select $\hat{y}_i \notin \mathbb{Z}$

$(\hat{x}^l,\hat{y}^l) \leftarrow \min(RMIP, y_i \leq \lfloor \hat{y}_i \rfloor)$

$(\hat{x}^r,\hat{y}^r) \leftarrow \min(RMIP, y_i \geq \lceil \hat{y}_i \rceil)$

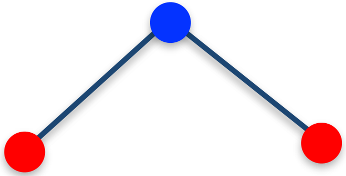
improve lower bound ⇨ $\min\{f(\hat{x}^l,\hat{y}^l), f(\hat{x}^r,\hat{y}^r)\} \geq f(\hat{x},\hat{y})$

Branch and Bound

- active node ●

Branch and Bound

- active node ●
- finished node ●

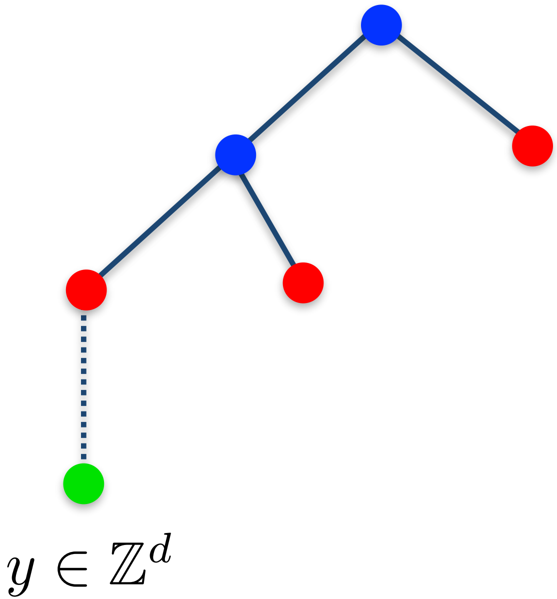


- lower bound: min{●}

Branch and Bound

- active node 🔴
- finished node 🔵
- feasible node 🟢

- lower bound: min{🔴}



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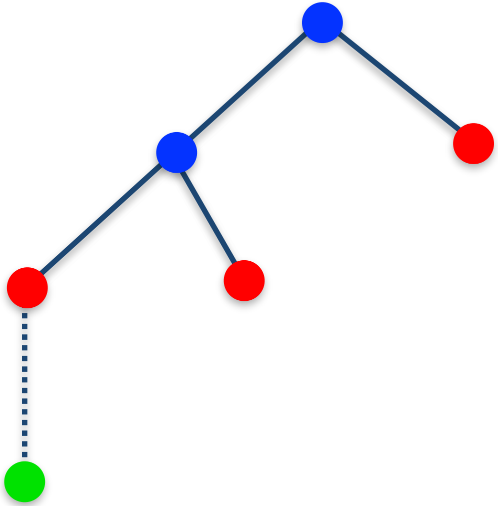

$y \in \mathbb{Z}^d$ ⇨ upper bound

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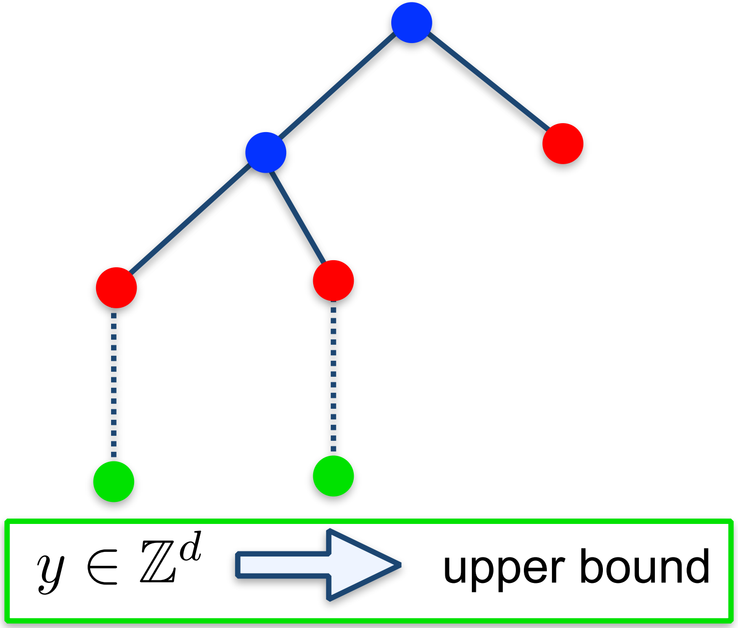

$y \in \mathbb{Z}^d$ → upper bound

Branch and Bound

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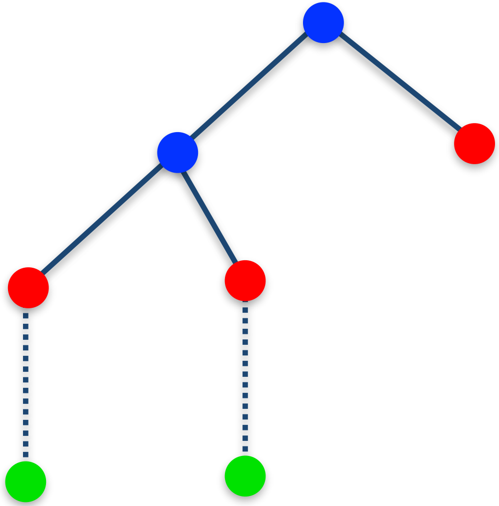

pruning
if 🔴 > upper bound
(or infeasible)

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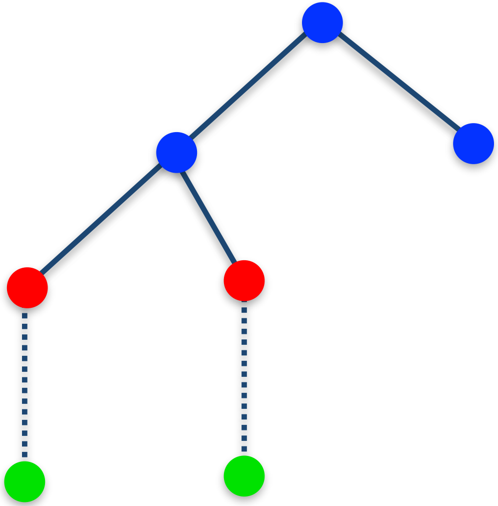

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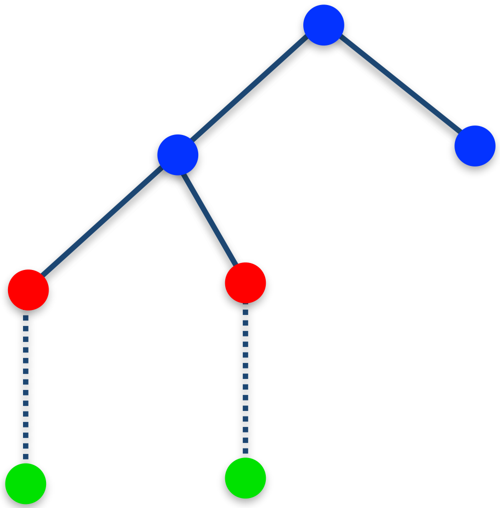


- lower bound: min{🔴}
- upper bound: min{🟢}

- terminate when min{🟢}-min{🔴} < epsilon

global optimum if RMIP convex!

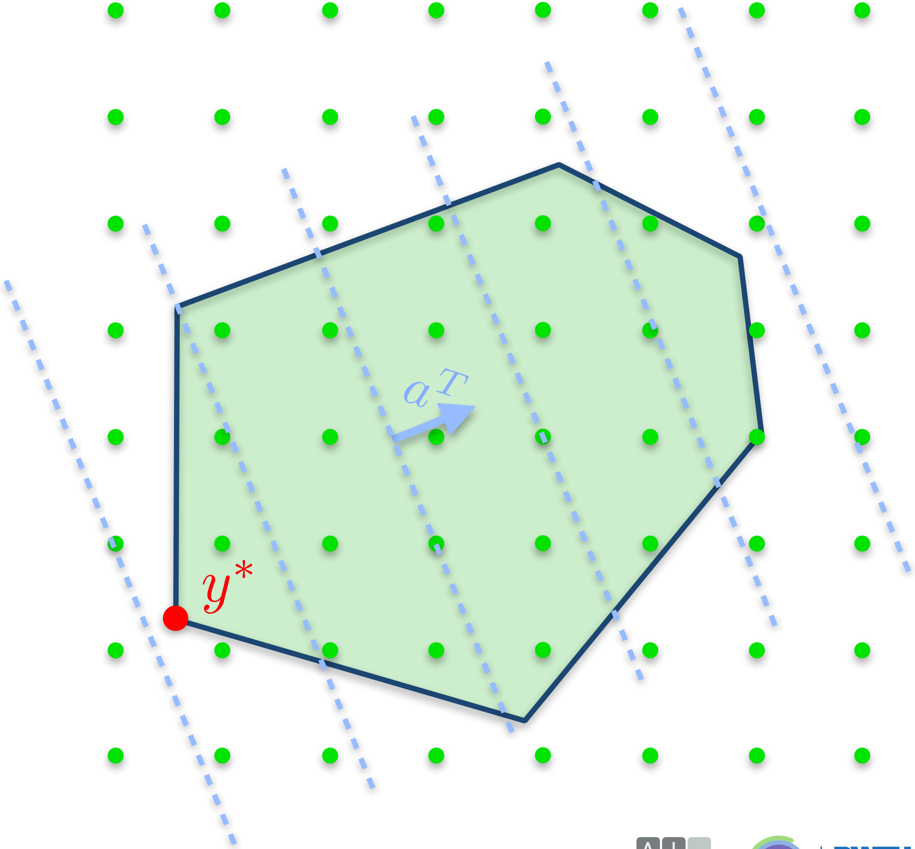
Branch and Bound

- Node Selection Strategies
  - Depth-First Search
  - Best-First Search (w.r.t. lower bound)
  - Dynamic heuristics

- Branching Strategies
  - Maximum Infeasibility — most undecided first (close to .5)
  - Strong Branching — best improvement of lower bound
  - Pseudo Reduced Cost — approximation of strong branching
  - Dynamic heuristics

Example: Branch and Bound

- Integer Linear Program (ILP)



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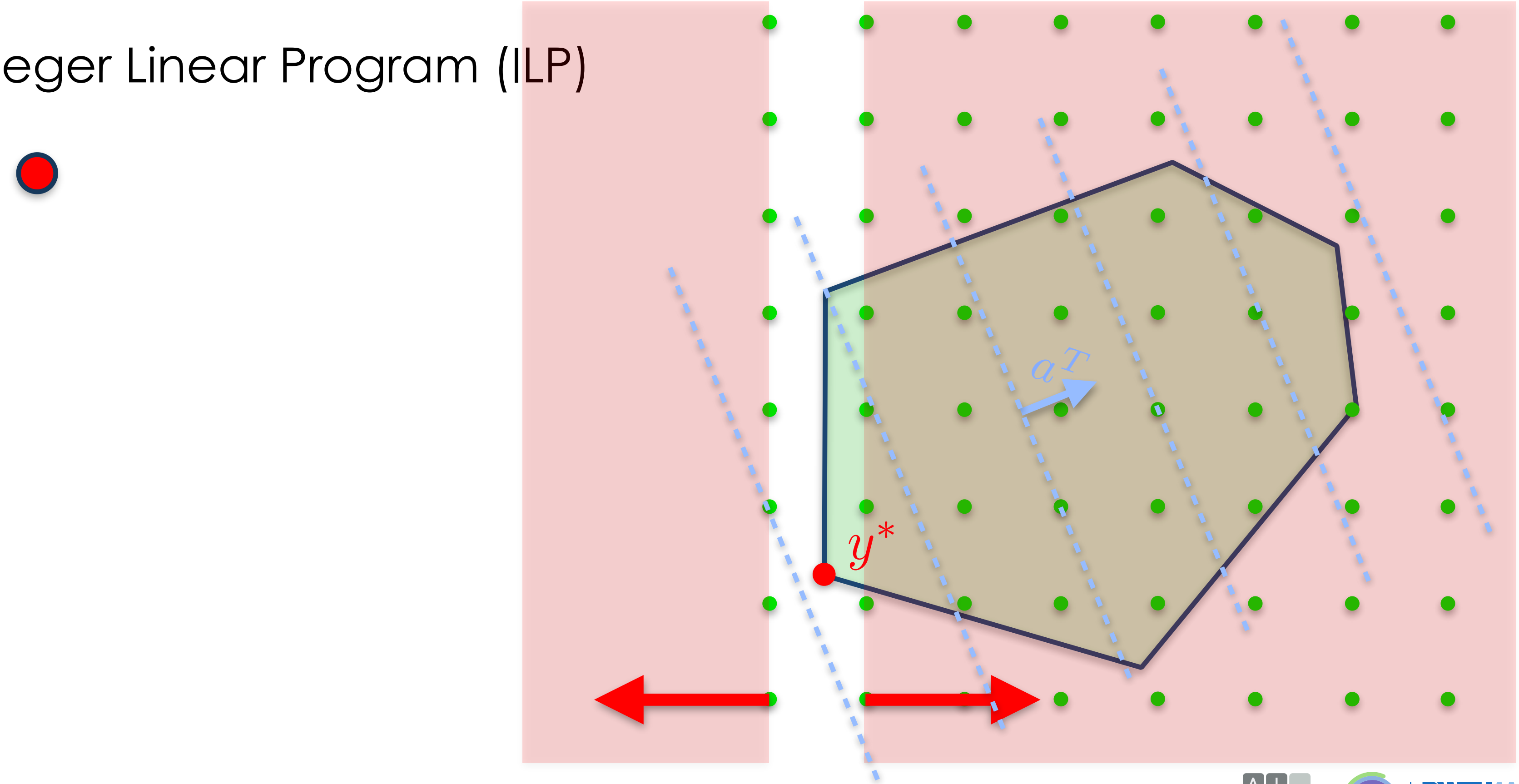

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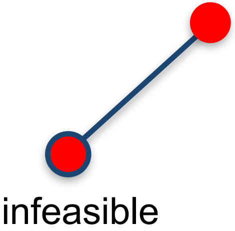

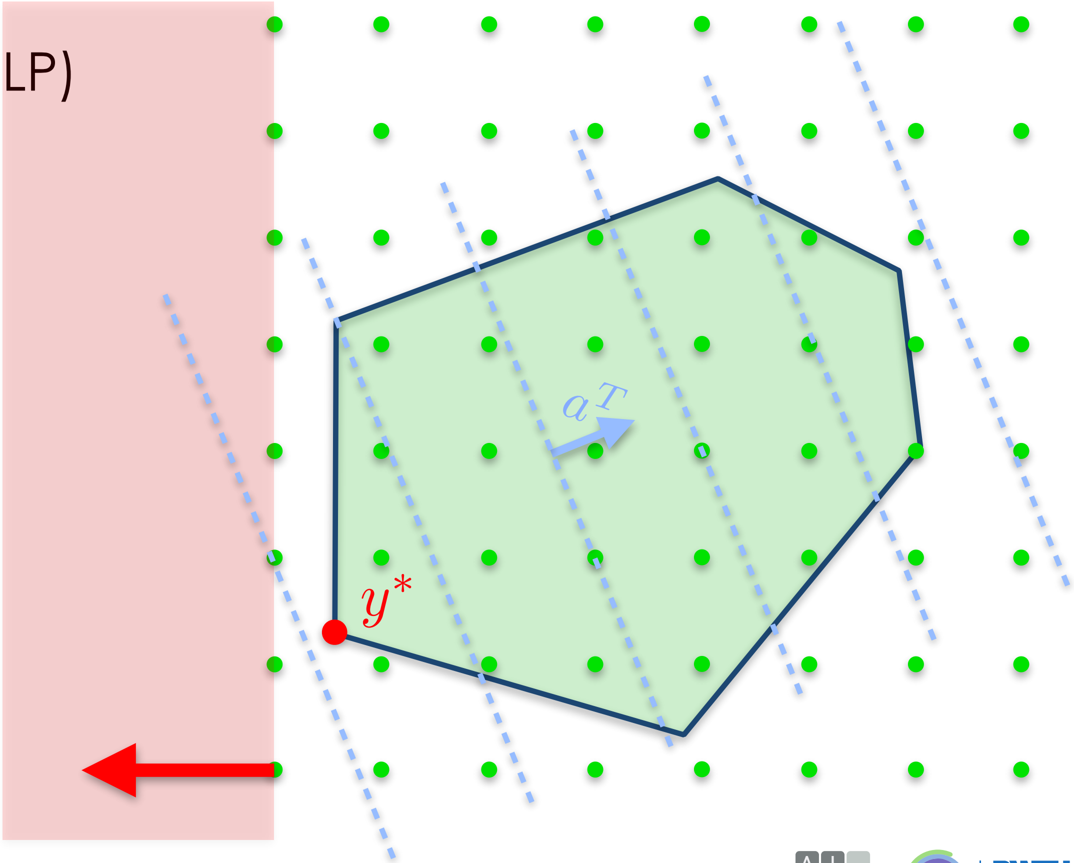

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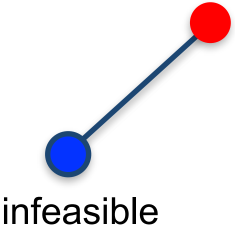

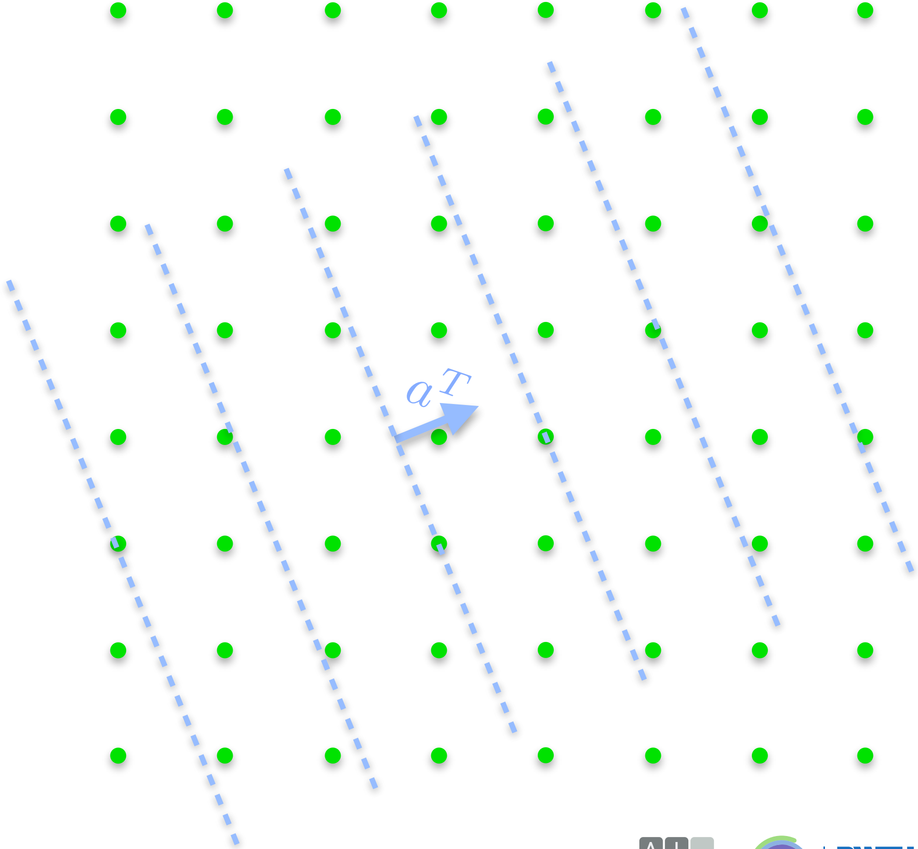

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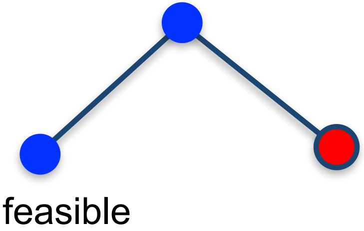

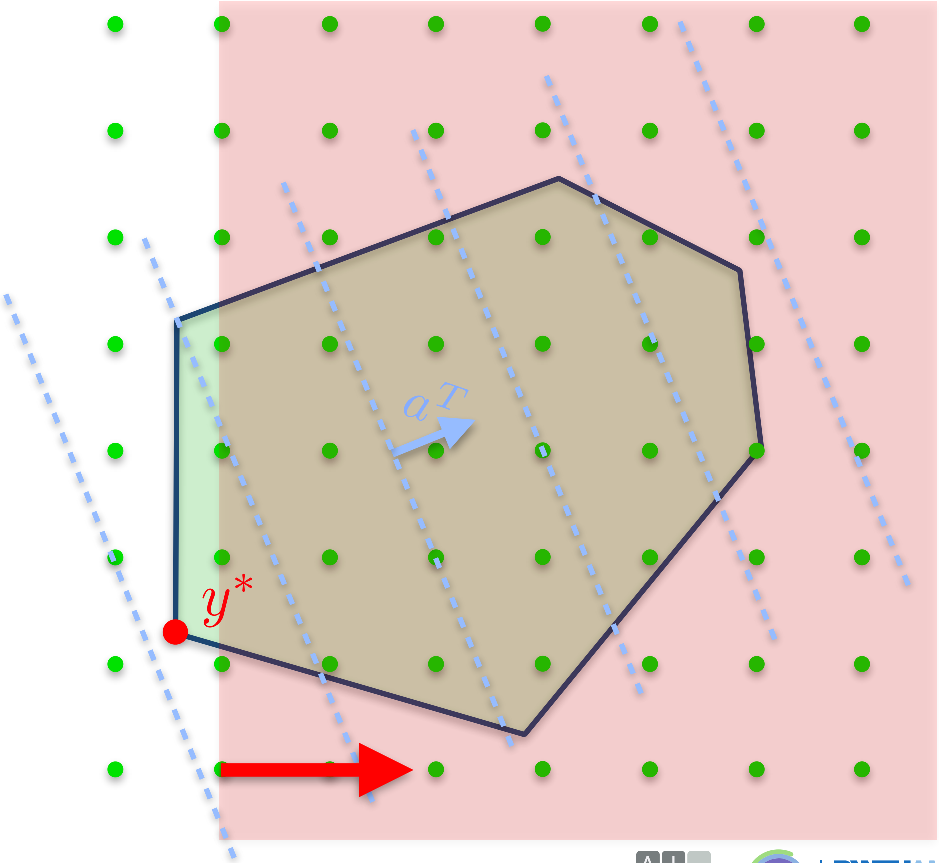

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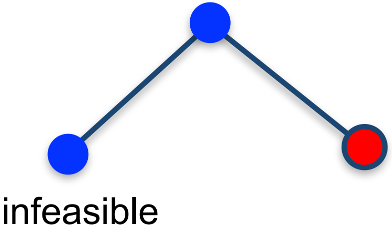

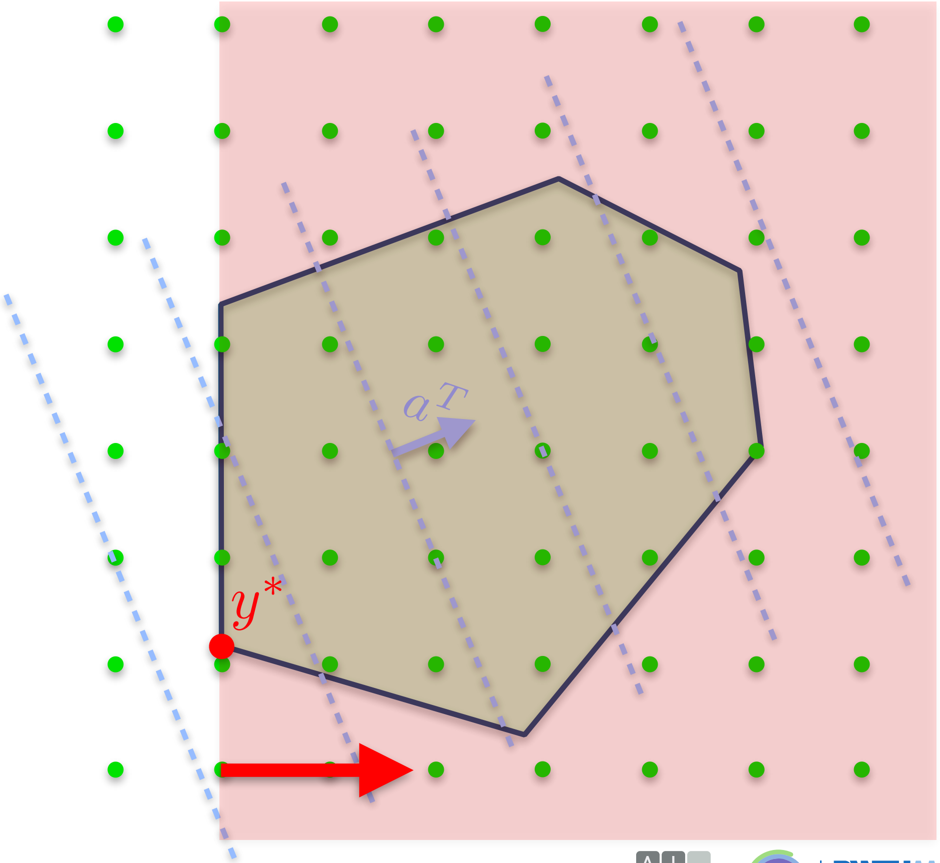

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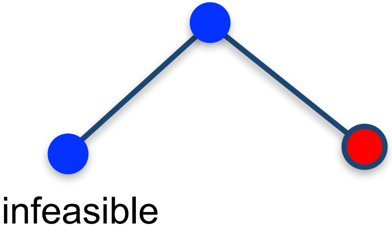

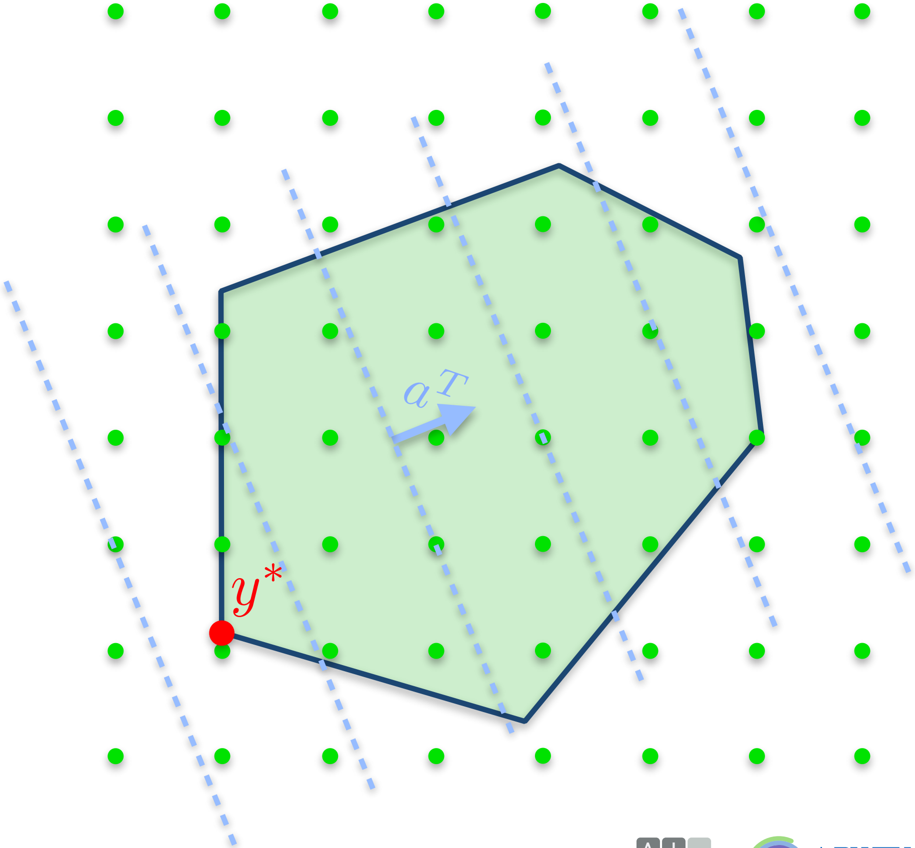

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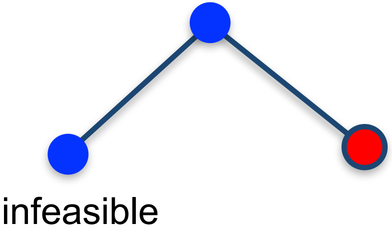

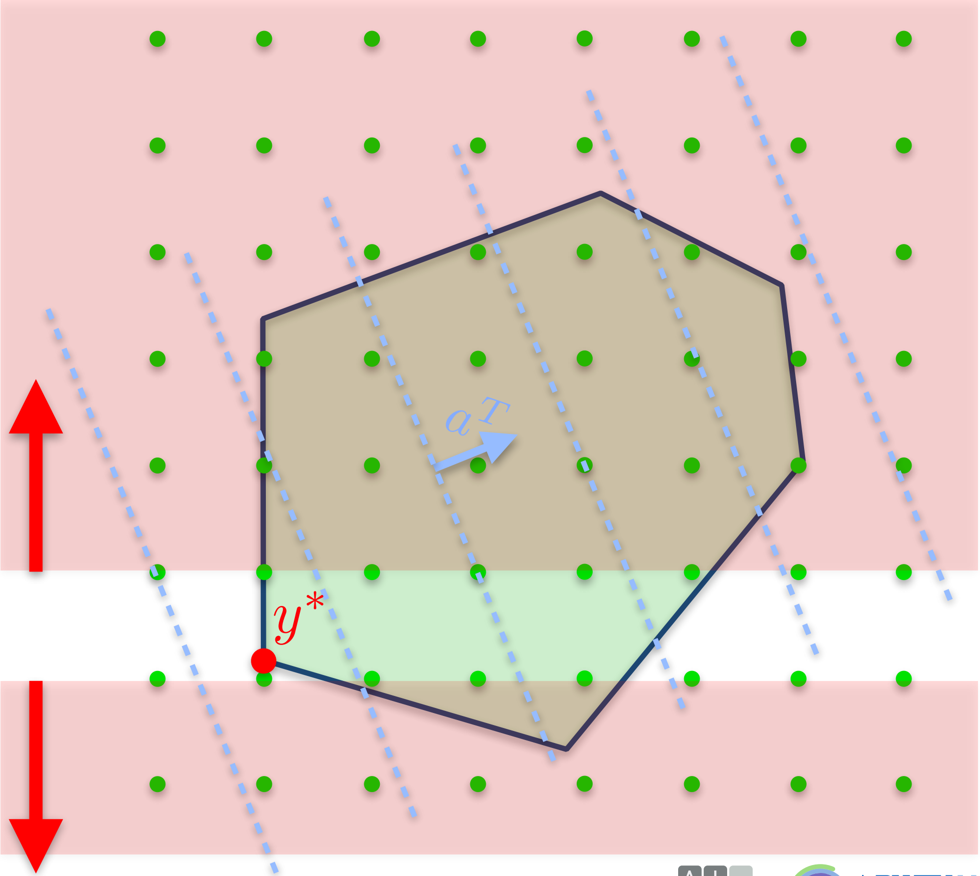

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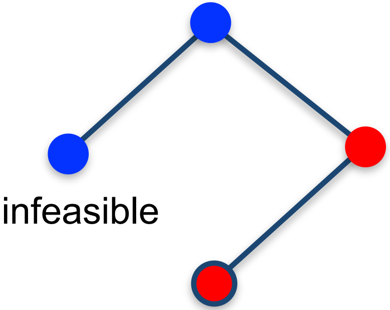

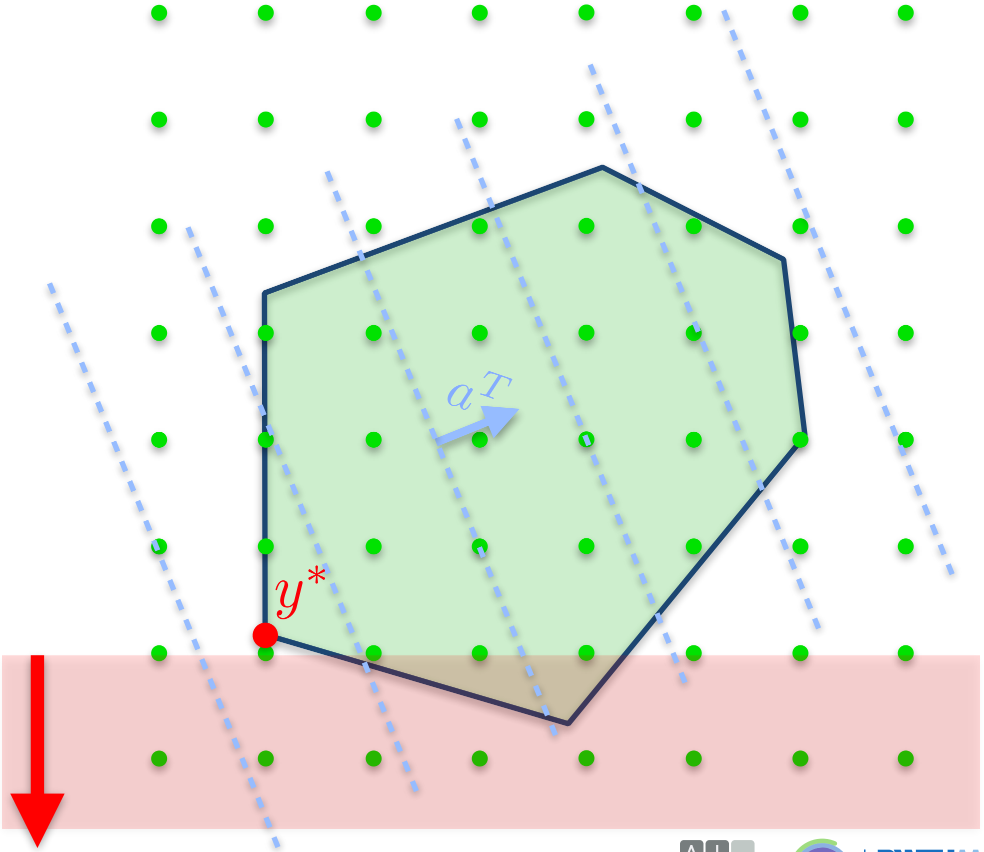

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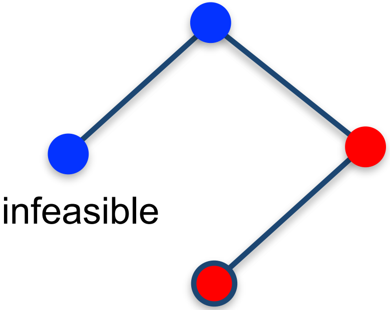

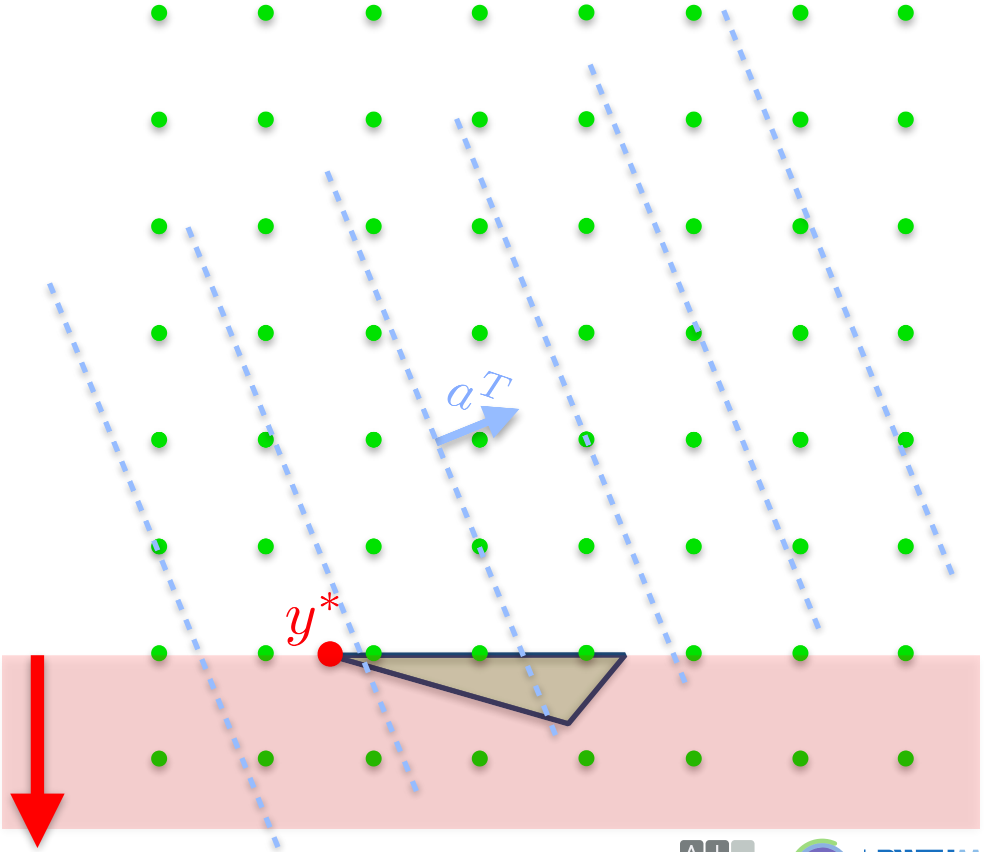

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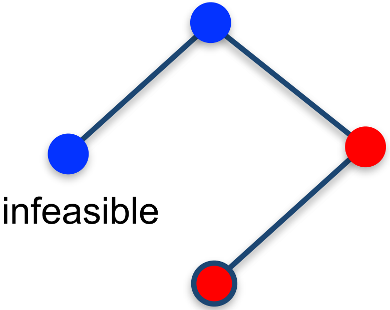

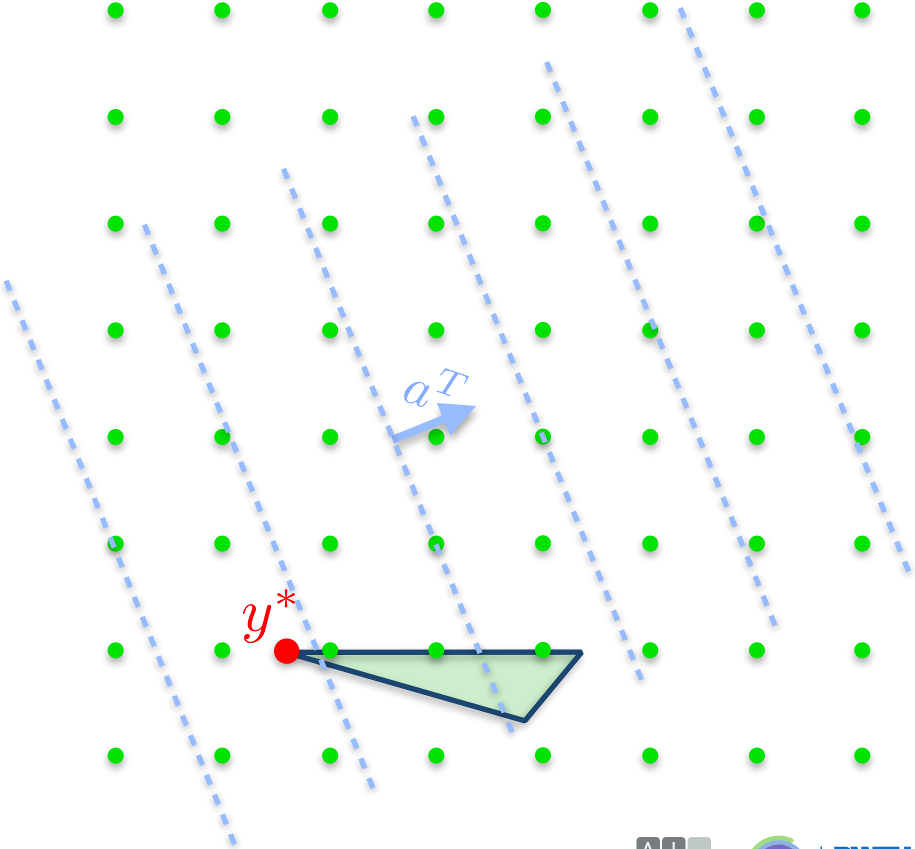

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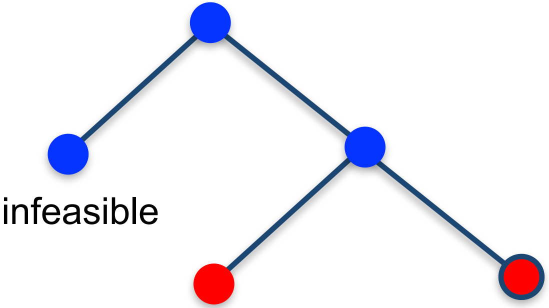
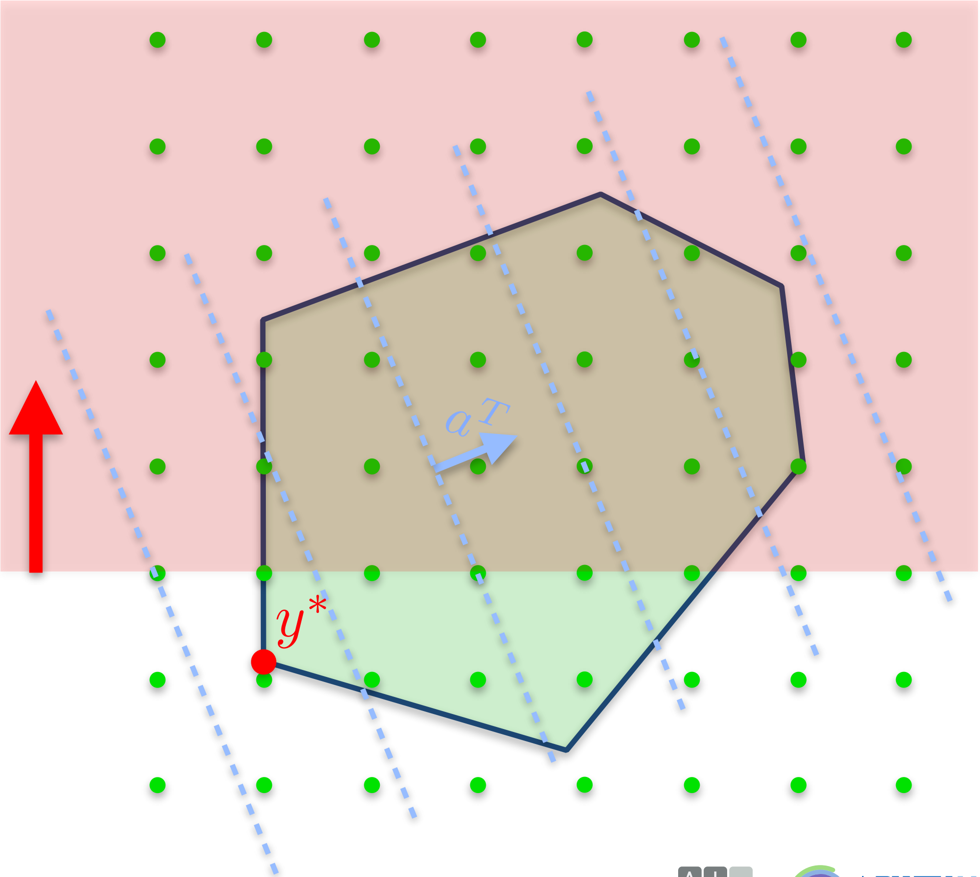
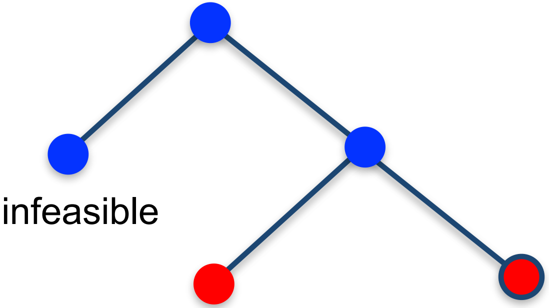
- Integer Linear Program (ILP)

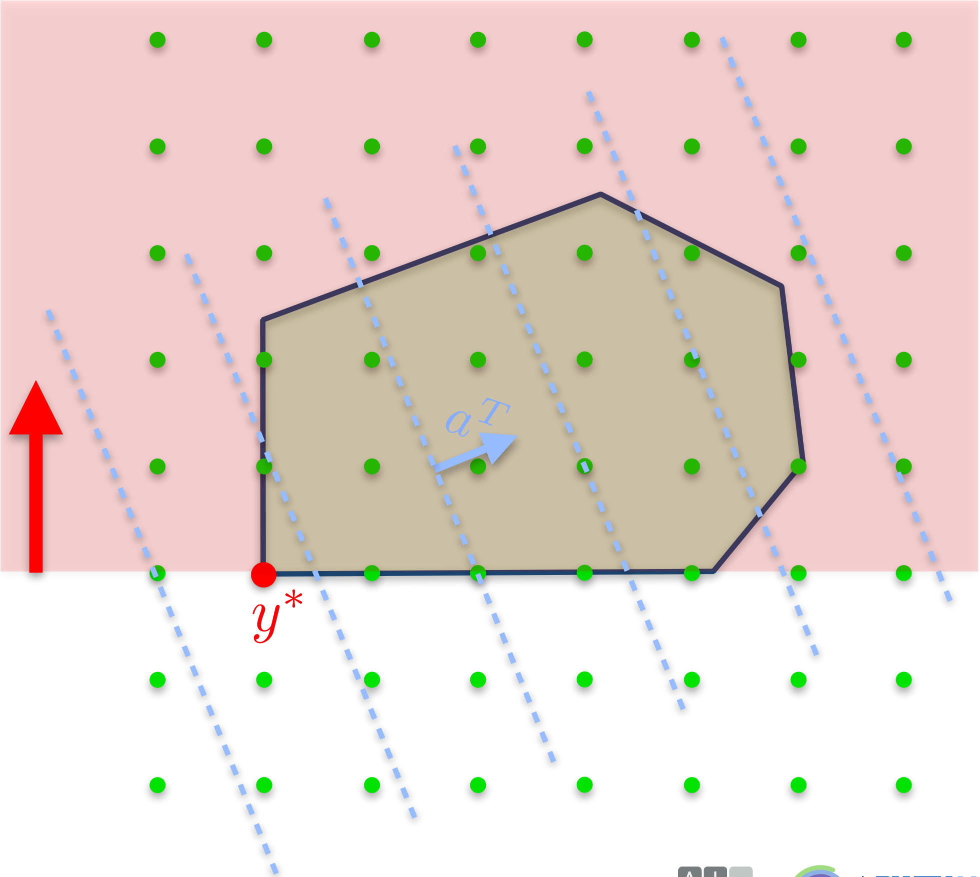




Example: Branch and Bound

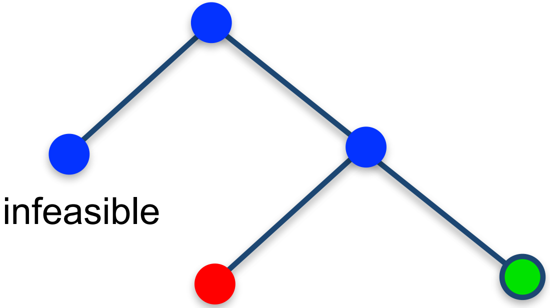
- Integer Linear Program (ILP)

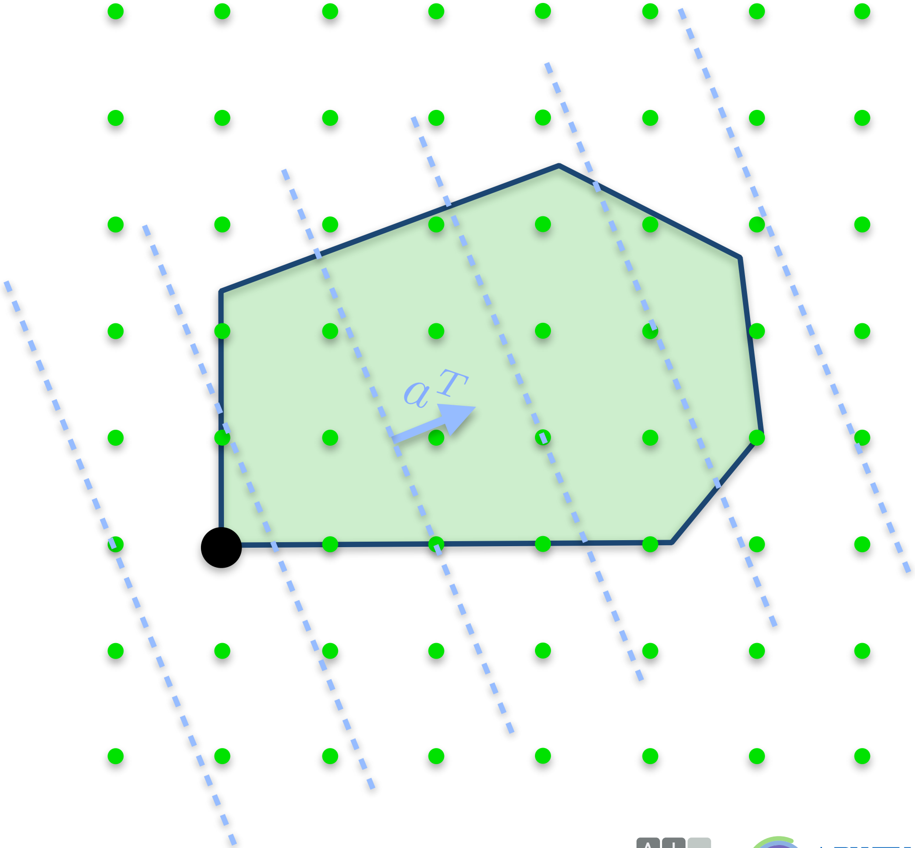




Example: Branch and Bound

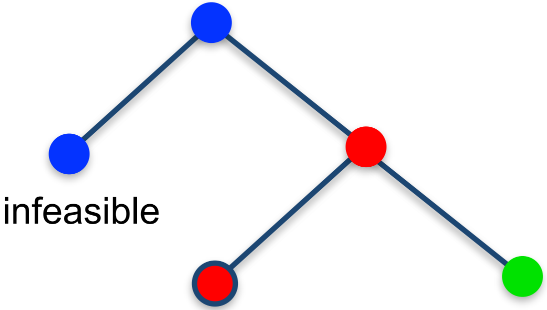
- Integer Linear Program (ILP)

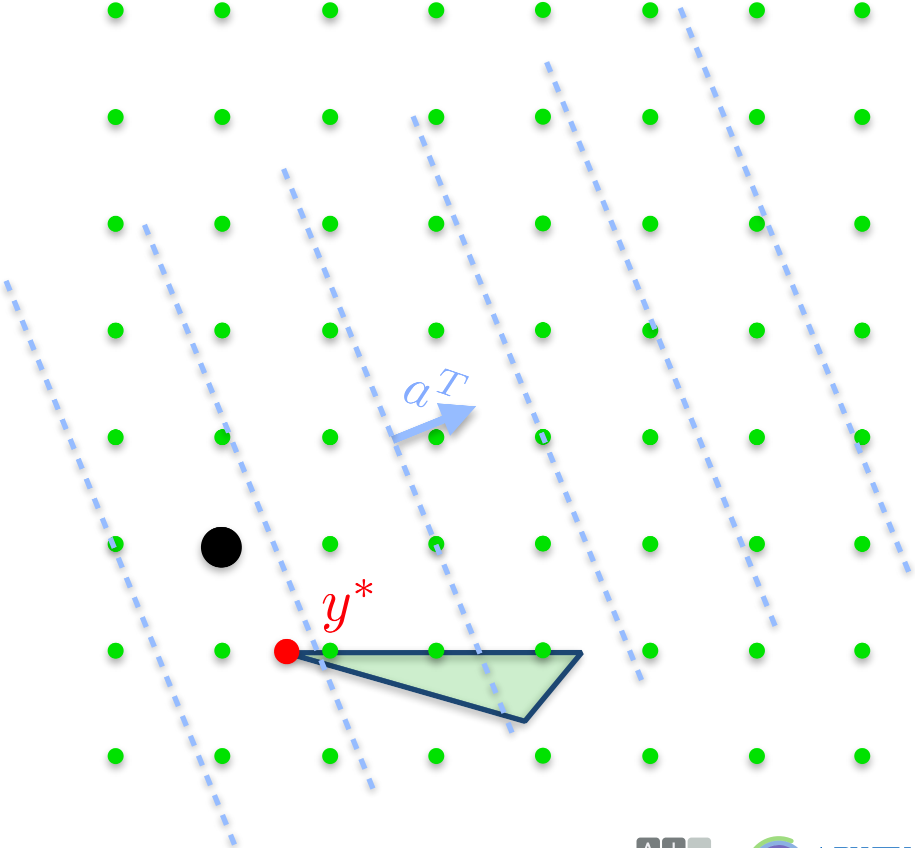




Example: Branch and Bound

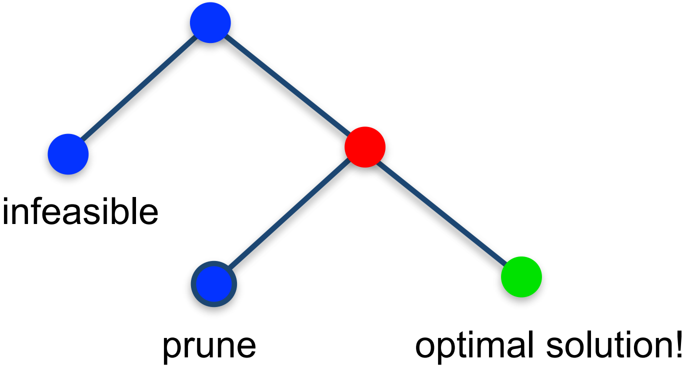
- Integer Linear Program (ILP)

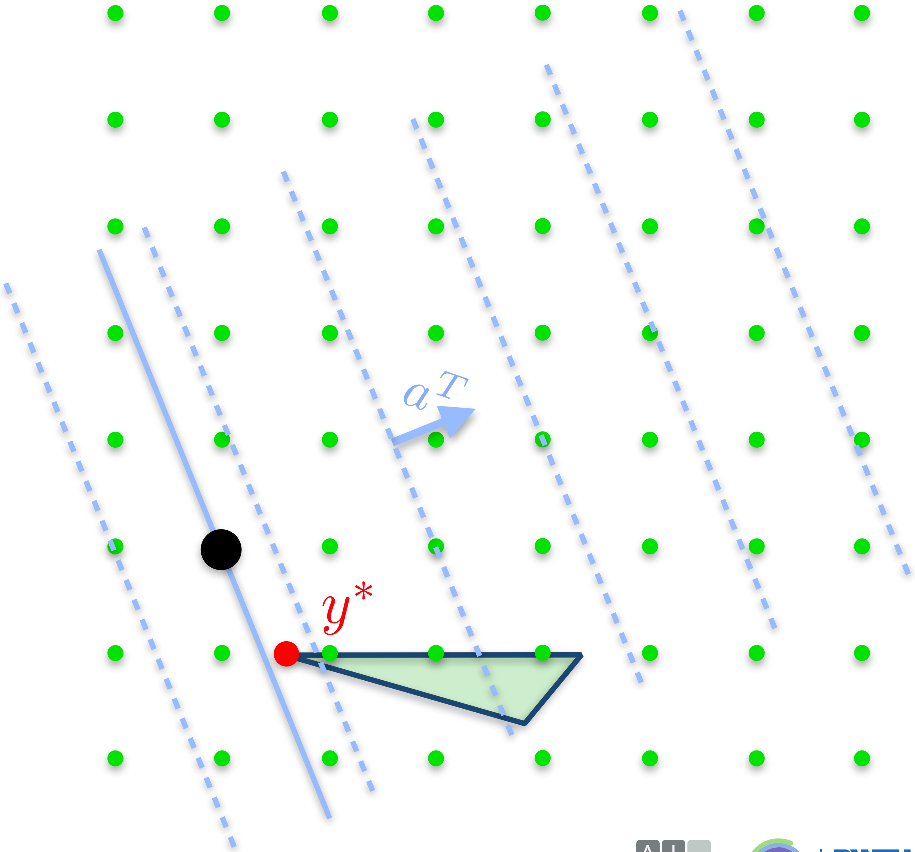




Example: Branch and Bound

- Integer Linear Program (ILP)



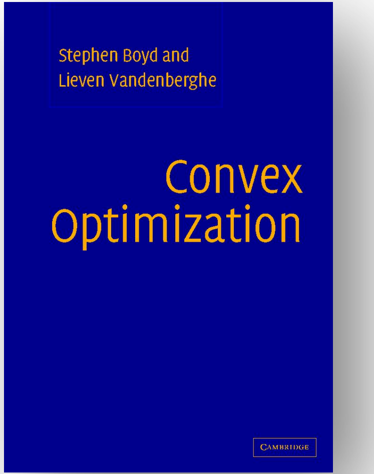


Summary

- Omnipresence of Optimization in Geometry Processing
- Unconstrained Optimization
- Equality Constrained Optimization
- Inequality Constrained Optimization
- Mixed-Integer Optimization
- Importance of Convexity

Outlook

- Further Reading



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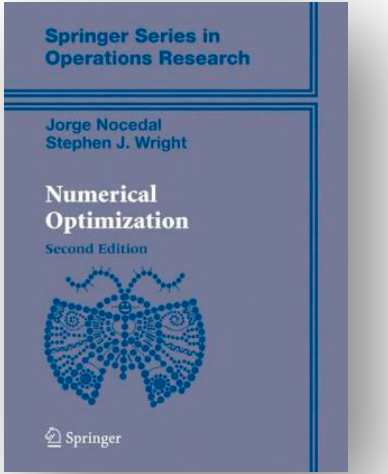

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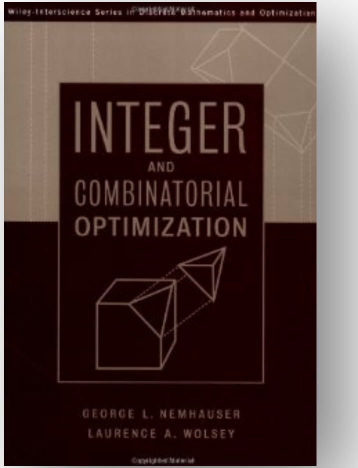

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Get PDF online:
http://stanford.edu/~boyd/cvxbook/

Software

- **Eigen** — linear algebra
- **IPOPT** — fast opensource C++ interior point method
- **Mosek** — commercial (convex) optimization in C, Java, Python…
- **Gurobi** — commercial mixed-integer optimization
- **CPLEX** — commercial mixed-integer optimization
- **Matlab** — many algorithms, good for prototyping
- **CVX** — prototyping for convex optimization
- **CoMISo** — unified interface to above algorithms

Thank You!