# Computational Aspects of Mappings

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# (boring!) Definition

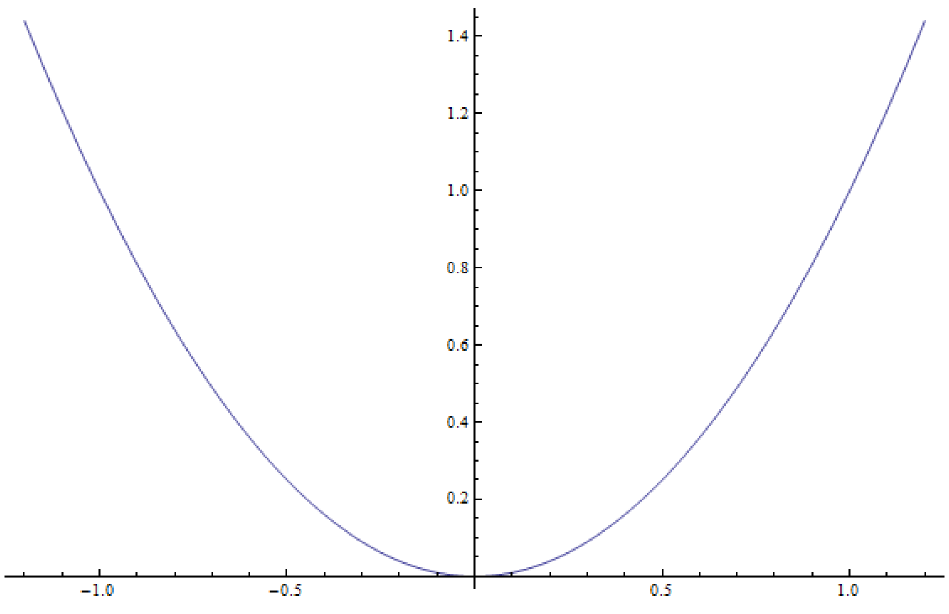
Mapping / Map :

A function between domains/ spaces

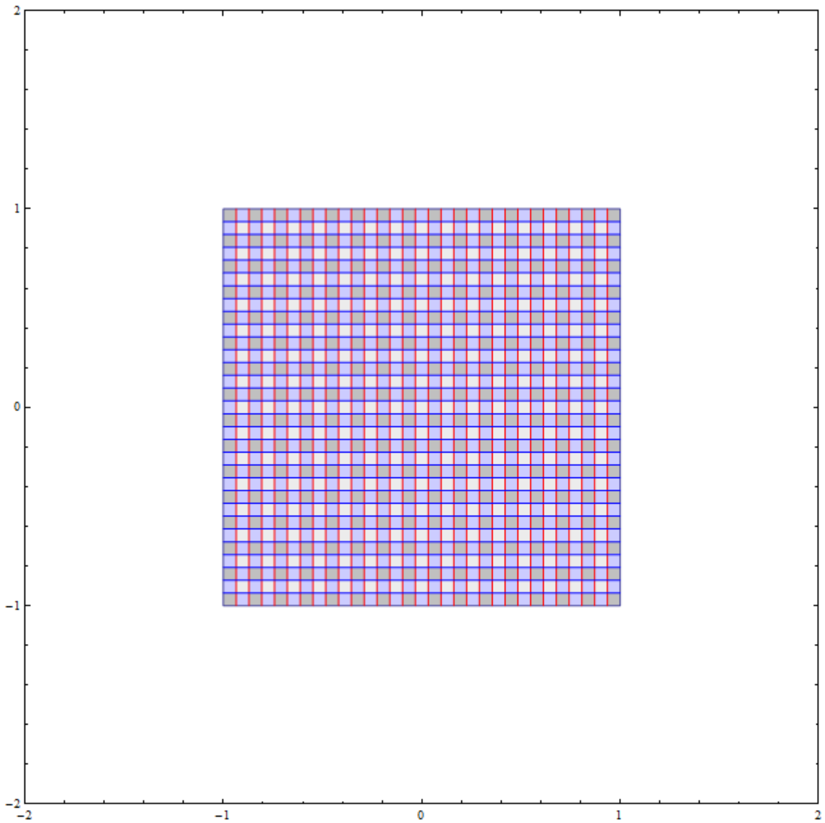
## Examples

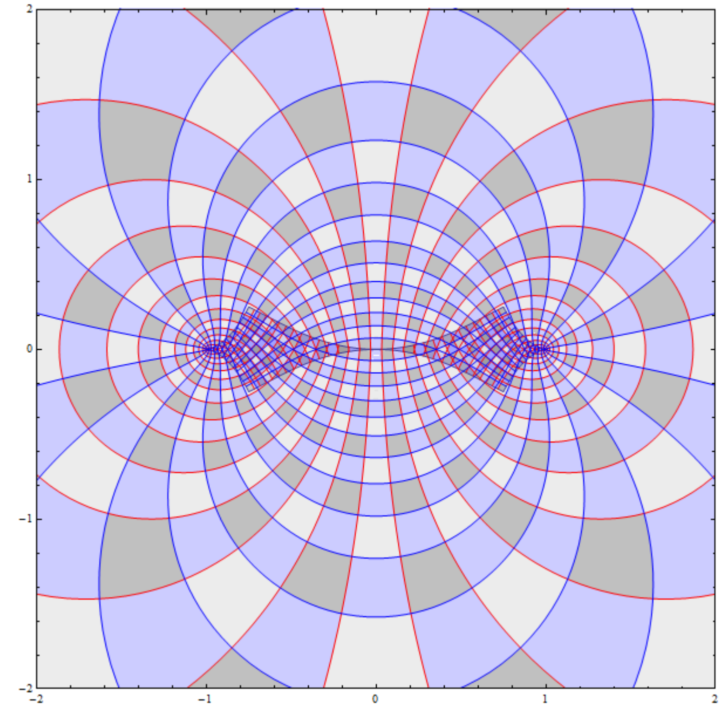
$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2$$

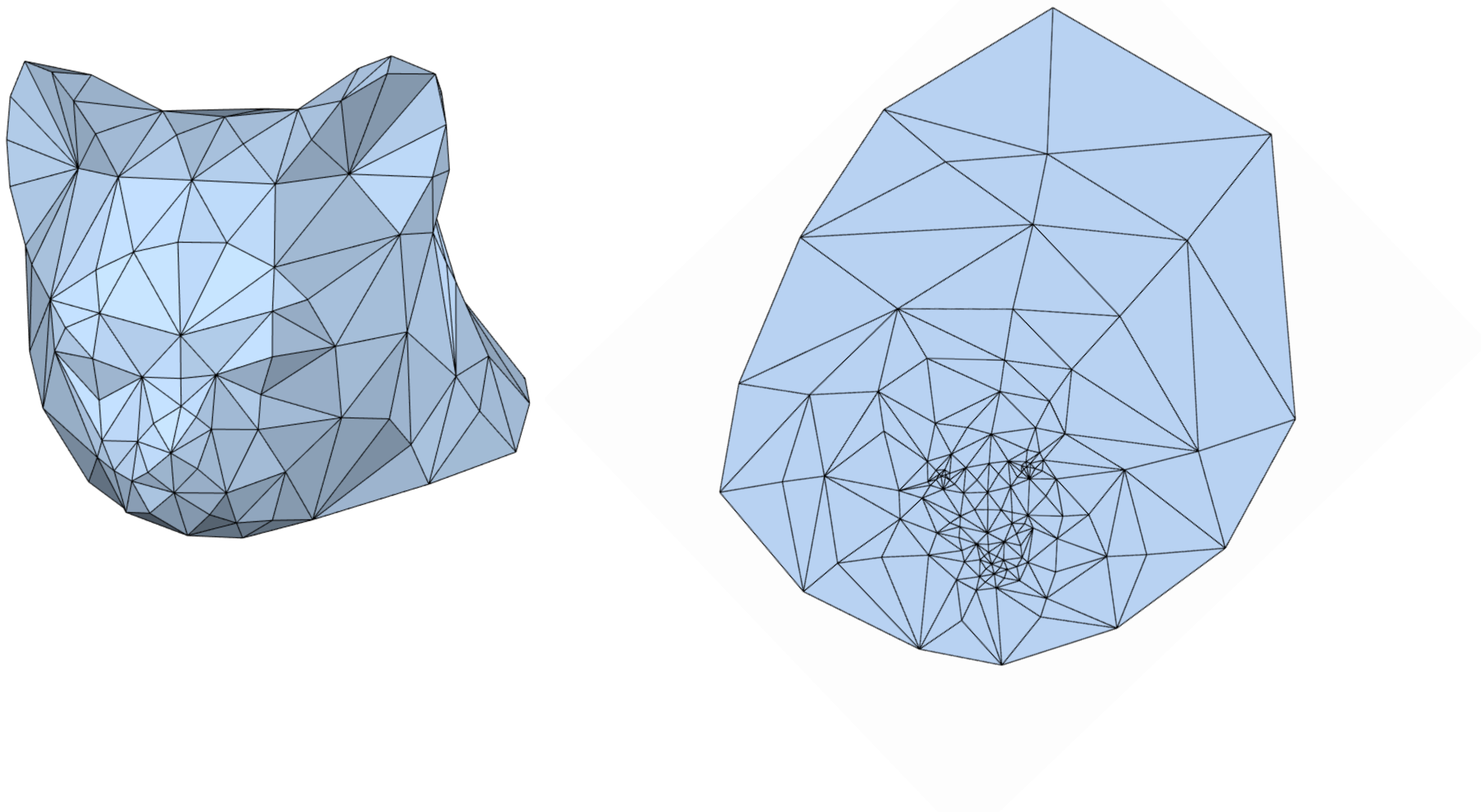


$$f: \mathbb{R}^2 \to \mathbb{R}^2$$



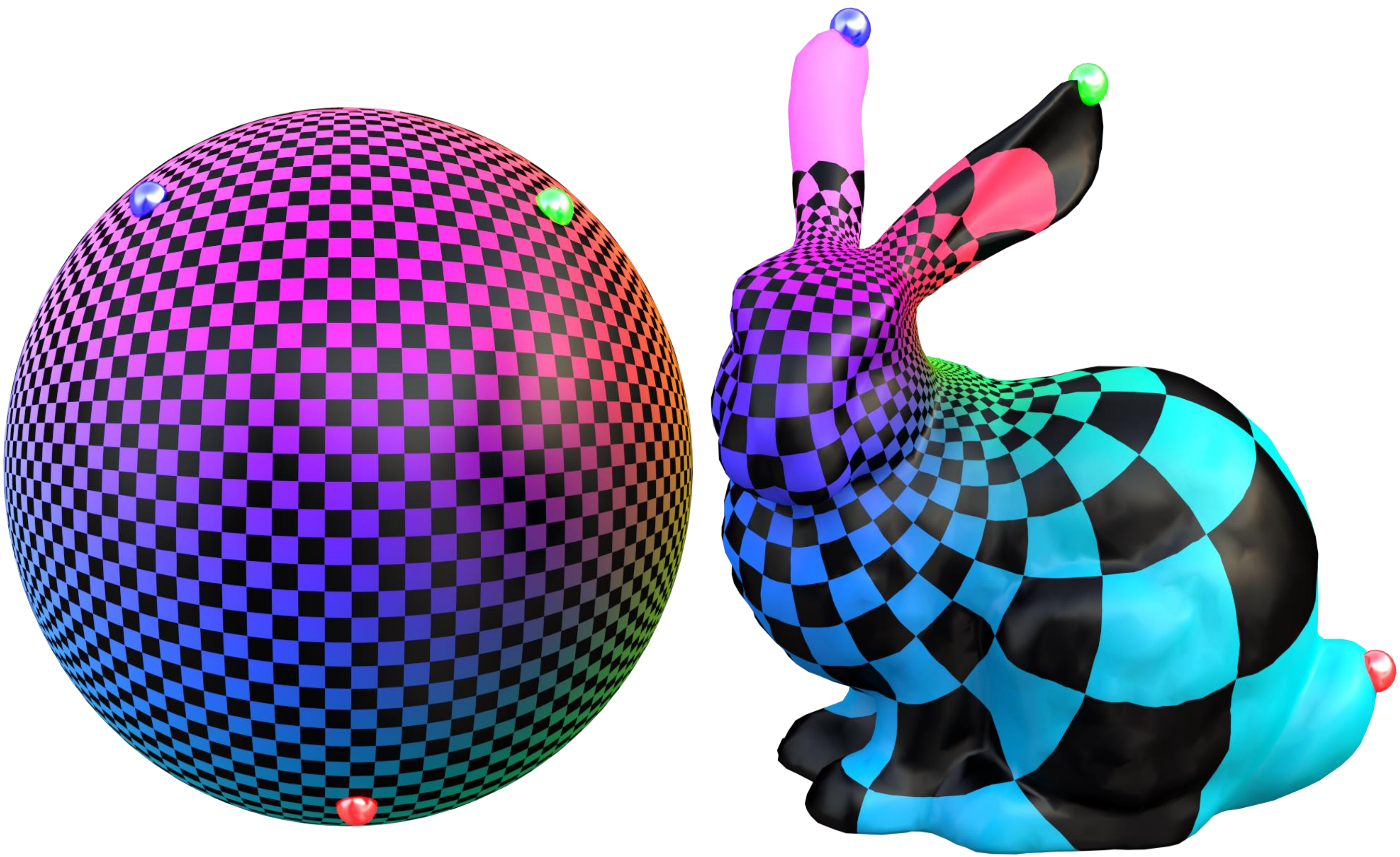



$$f: \mathbb{M} \rightarrow \mathbb{R}^2$$

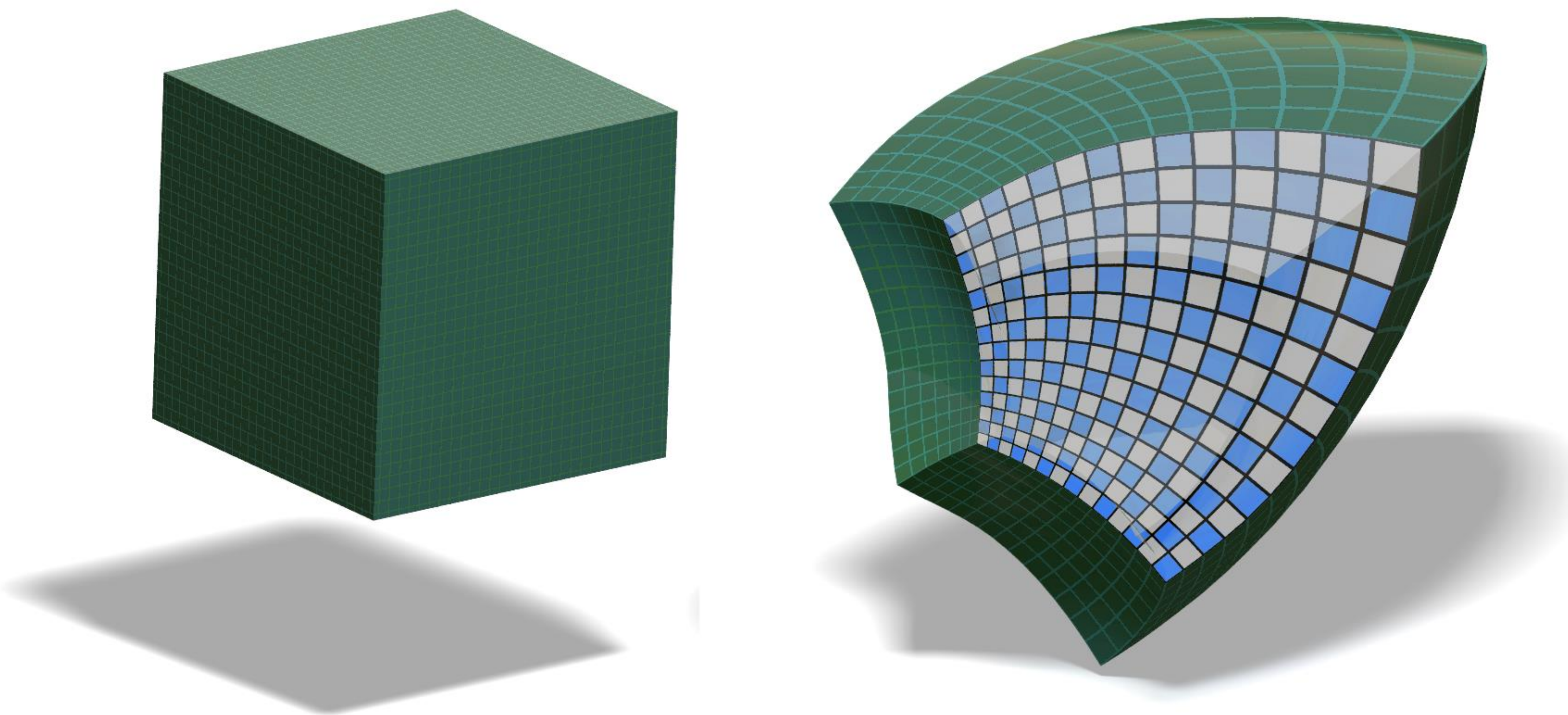


$$f: \mathbb{M} \to \mathbb{M}'$$

$f: \mathbb{S}^2 \to \mathbb{M}$

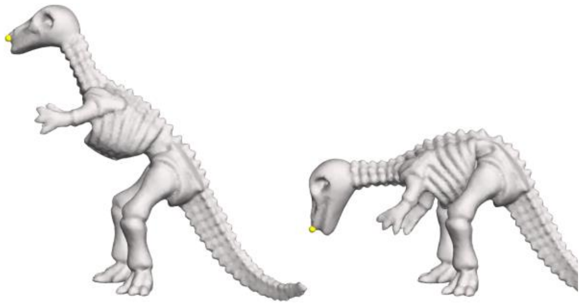


$$f : V \to \mathbb{R}^3$$

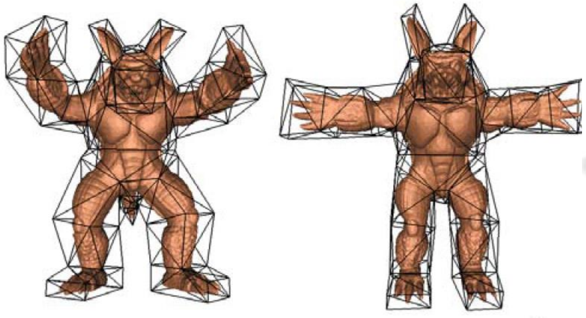


# Applications…

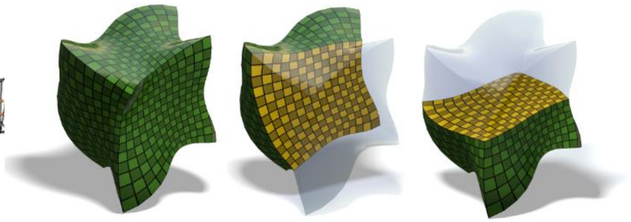
# Deformations



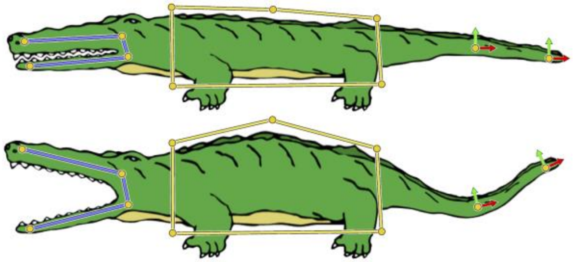
[Sorkine & Alexa 07]



[Ju et al. 05]

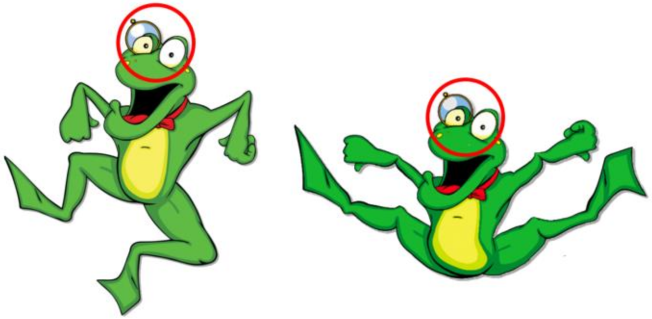


[Kovalsky et al. 2014]



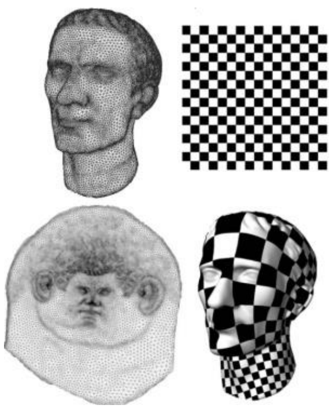
[Jacobson 07]

[Weber et al. 09]

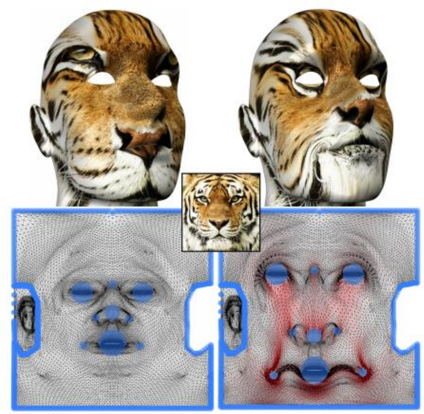


[Lipman et al. 07]

# Parameterizations

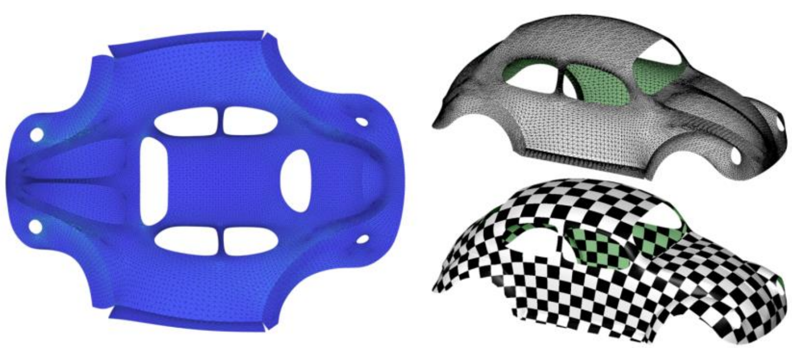


[Lévy et al. 02]

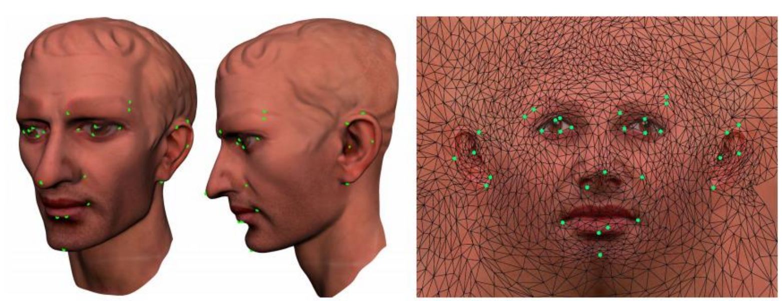


[Schuler et al. 13]

[Fu et al. 15]



[Mullen et al. 08]

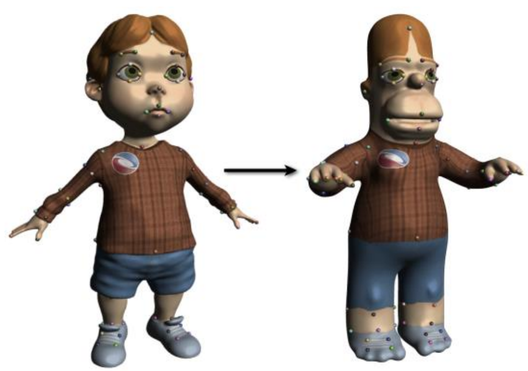


[Weber et al. 12]

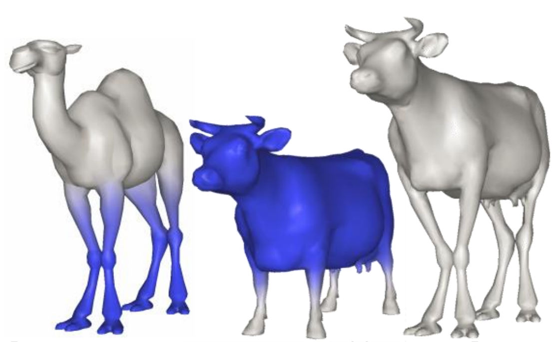
# Surface mappings

[Kim et al. 11]

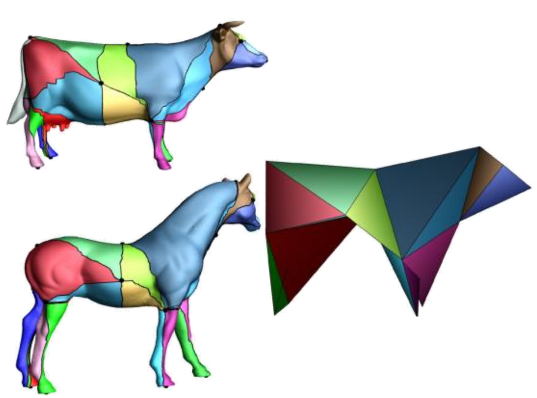
[Ovsjanikov et al. 12]



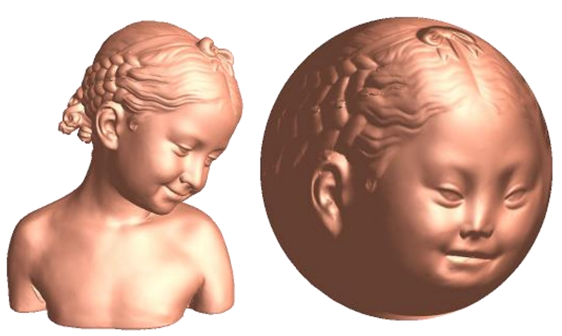
[Panozzo et al. 13]



[Kraevoy and Sheffer 04]



[Schreiner et al. 04]

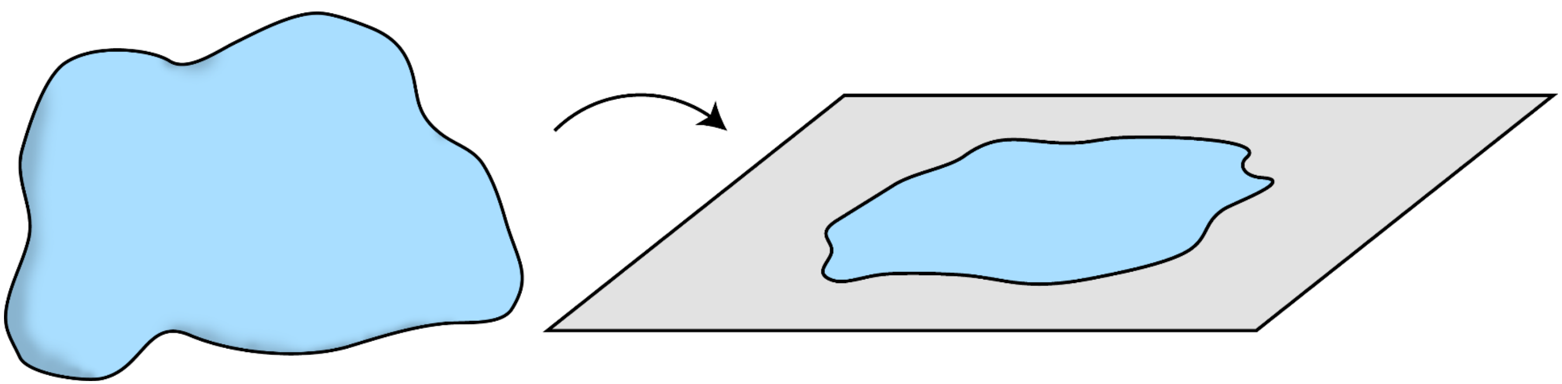


[Jin et al. 08]

# Discrete maps…

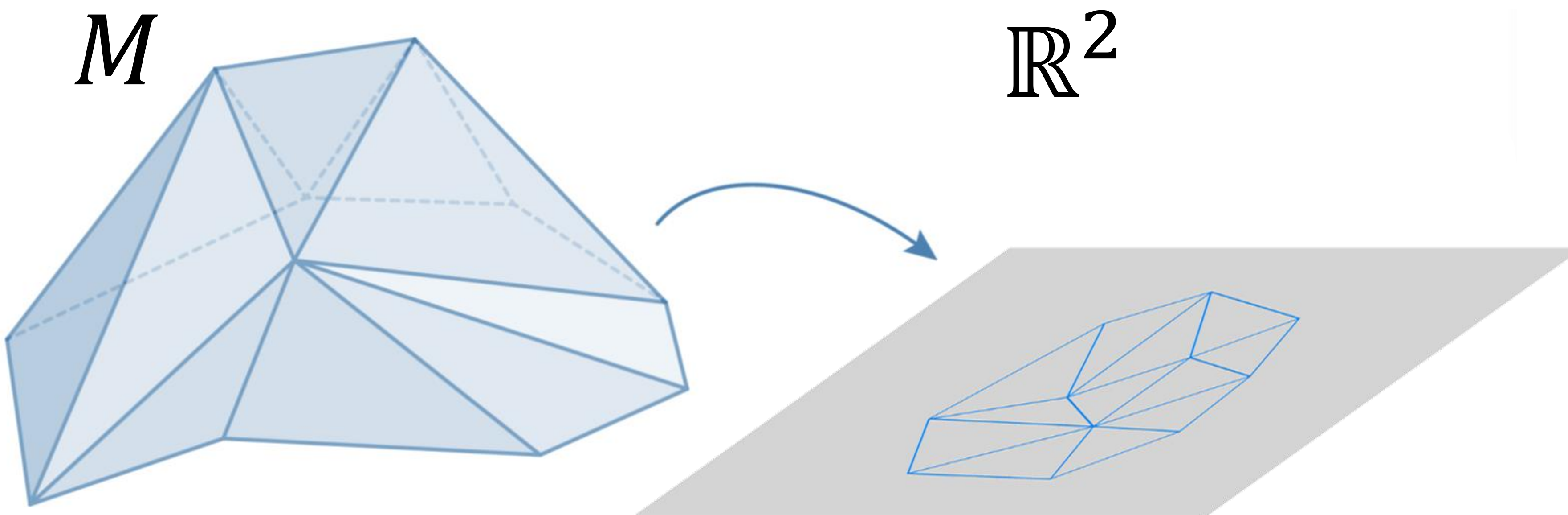
# Smooth case

e.g., a smooth surface is mapped to $\mathbb{R}^2$

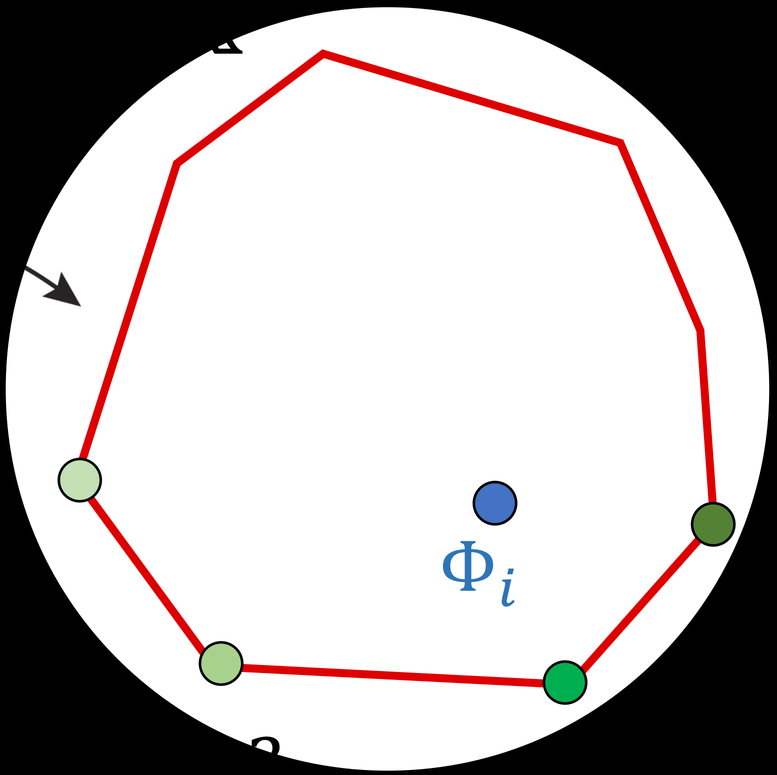


# Discrete case

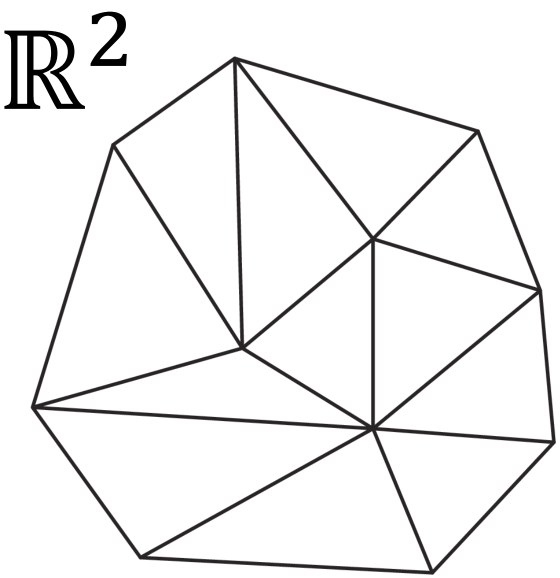
e.g., a surface **mesh** is mapped to $\mathbb{R}^2$



Focus on a
Specific embedding

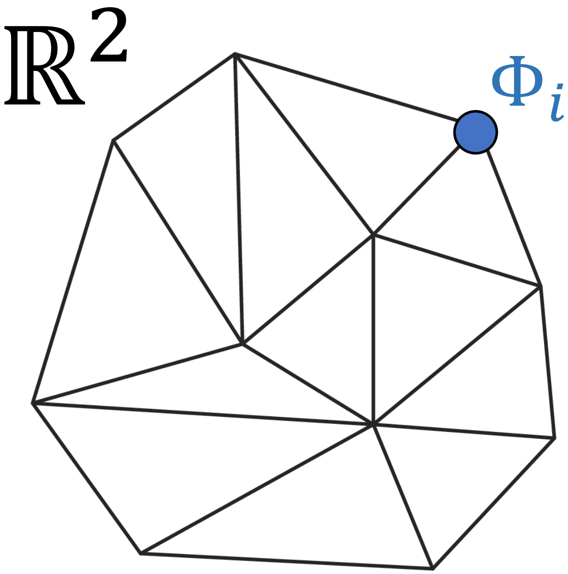


# Tutte's embedding



# Tutte's embedding

1. Boundary mapped to convex polygon

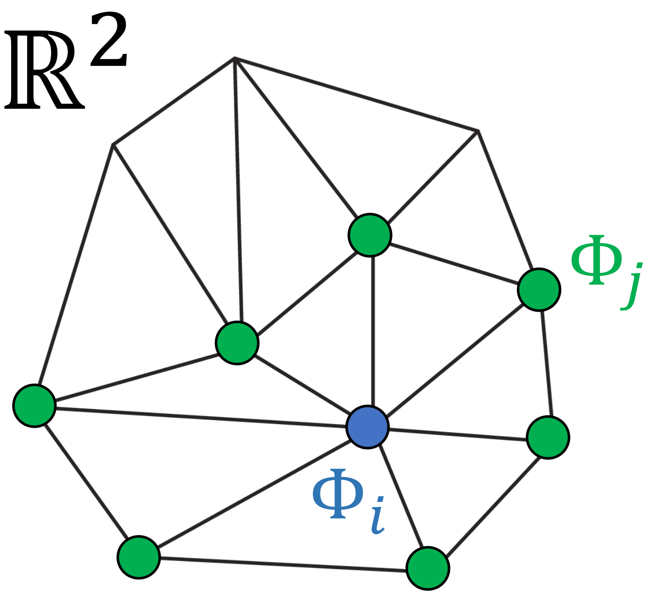



Linear in $\Phi_i$

$$\Phi_{\mathrm{i}} = \mathrm{p}_{\mathrm{i}}$$

# Tutte's embedding

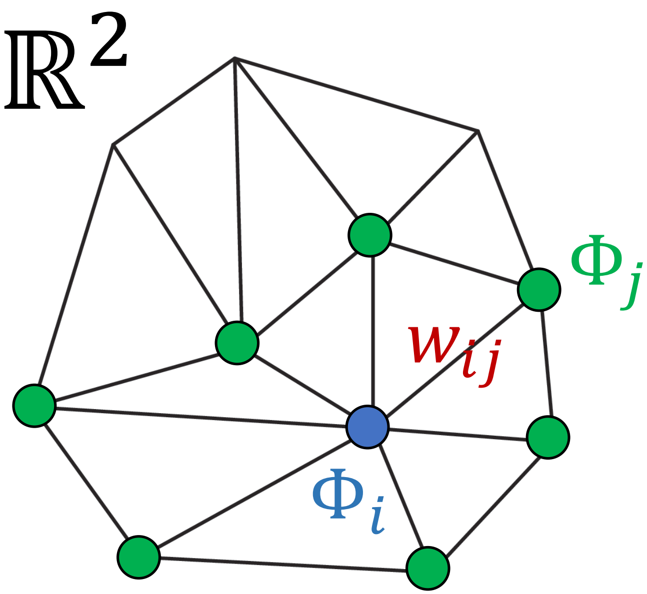
1. Boundary constrained to convex polygon
2. **Discrete Harmonic** - Interior vertices at average of neighbors




$$\Phi_i = \frac{1}{|N_i|} \sum_{j \in N_i} \Phi_j$$

# Tutte's embedding

1. Boundary constrained to convex polygon
2. **Discrete Harmonic** - Interior vertices at average of neighbors




Linear in $\Phi_i$

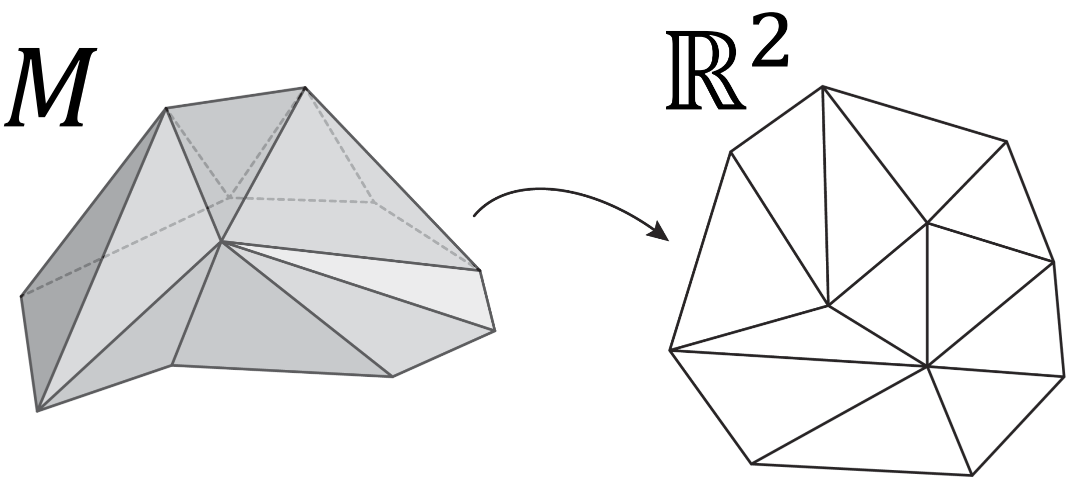
$$\Phi_i = \sum_{j \in N_i} w_{ij} \Phi_j$$

($W_{ij}$ positive, sum to 1)

# Tutte's embedding

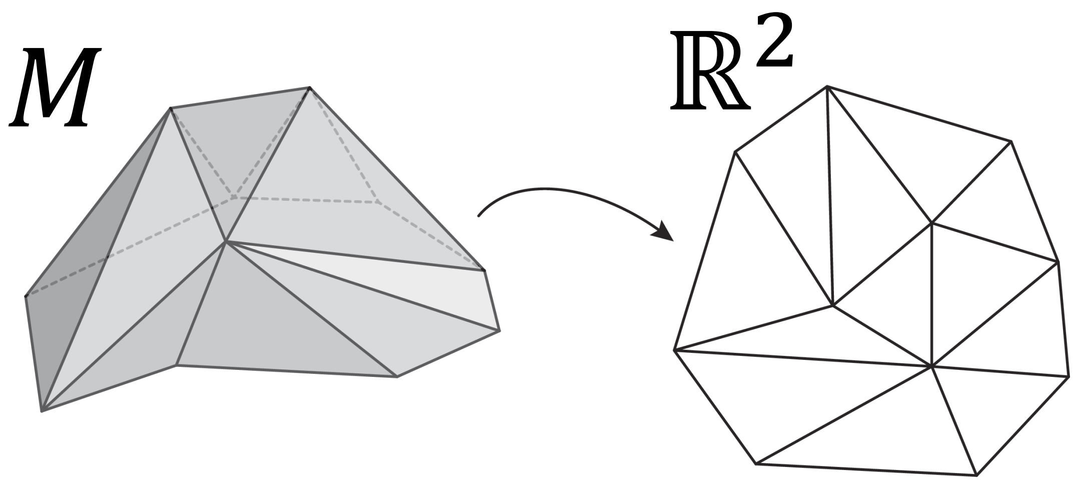
1. Boundary to convex polygon
2. Harmonic

Solve sparse linear system!



# Tutte's embedding

1. Bijective: The graph edges don't overlap themselves
2. Discrete Harmonic: analog to smooth harmonic maps



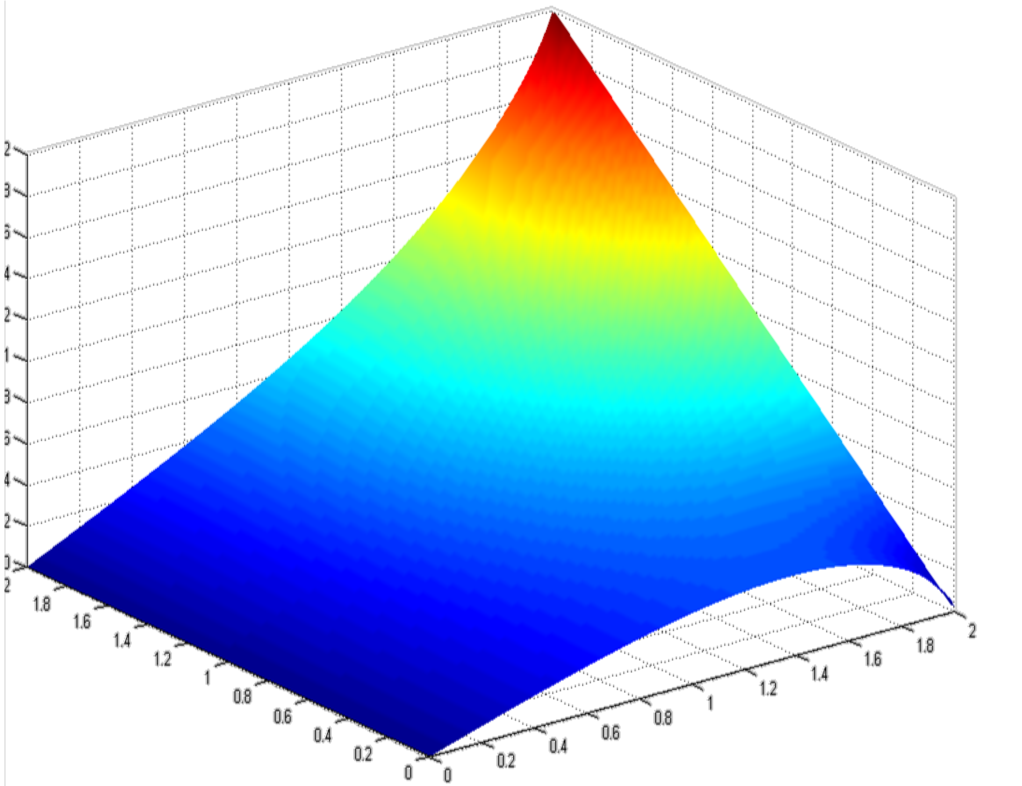
# Continuous harmonic maps

- Harmonic **function**:

$$\Delta f = 0$$

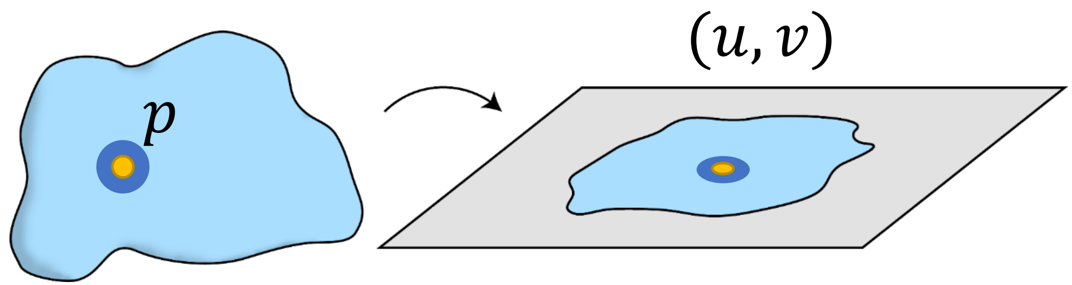
**"Laplacian – difference of value at point to average of neighborhood"**

$$z = f(x, y)$$

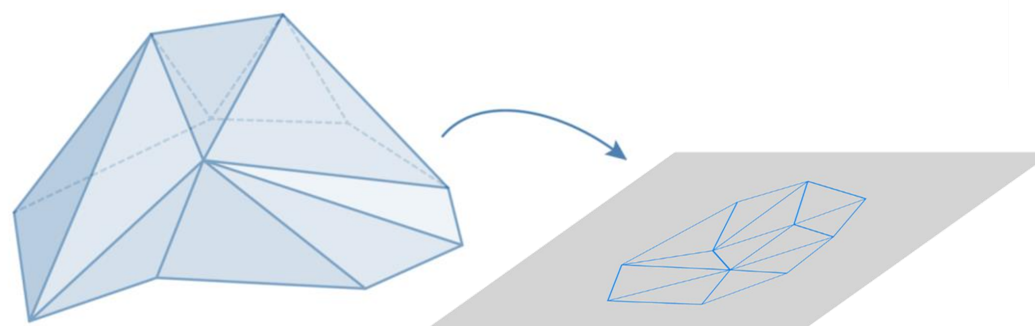


# Continues harmonic maps

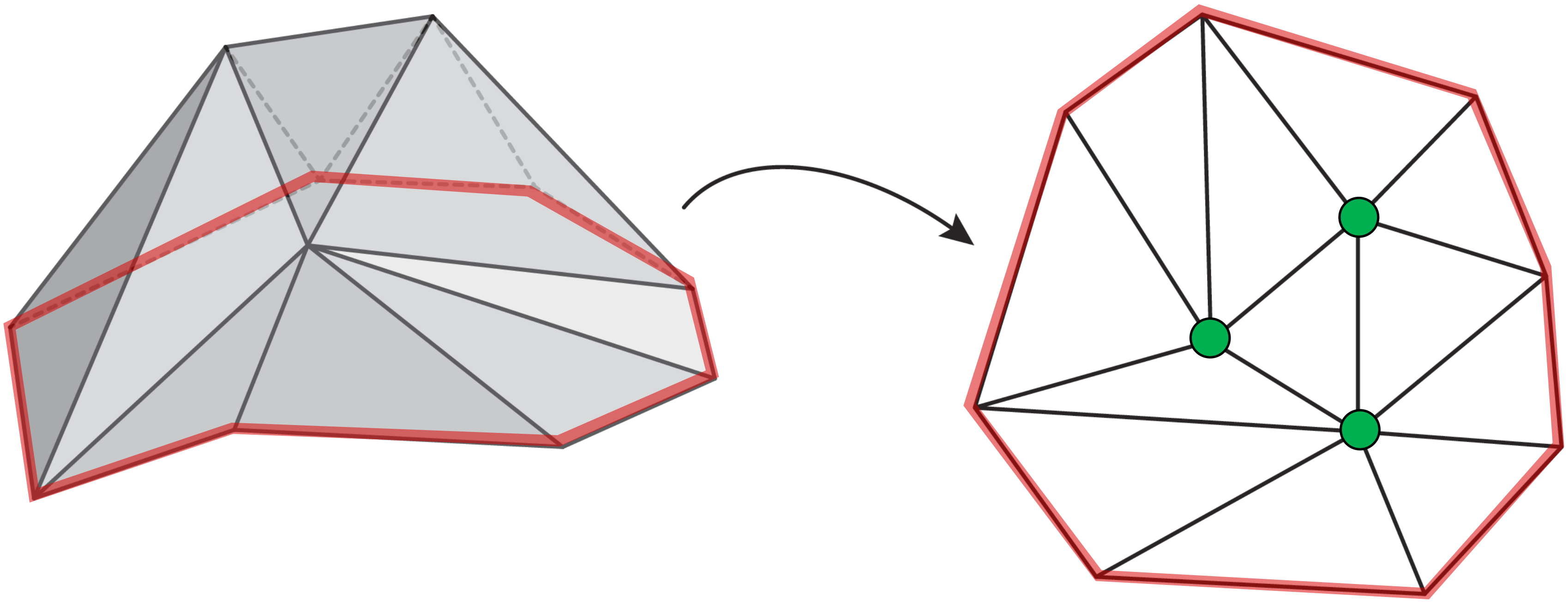
- Harmonic **map**:
$$\Delta u = 0$$
$$\Delta v = 0$$



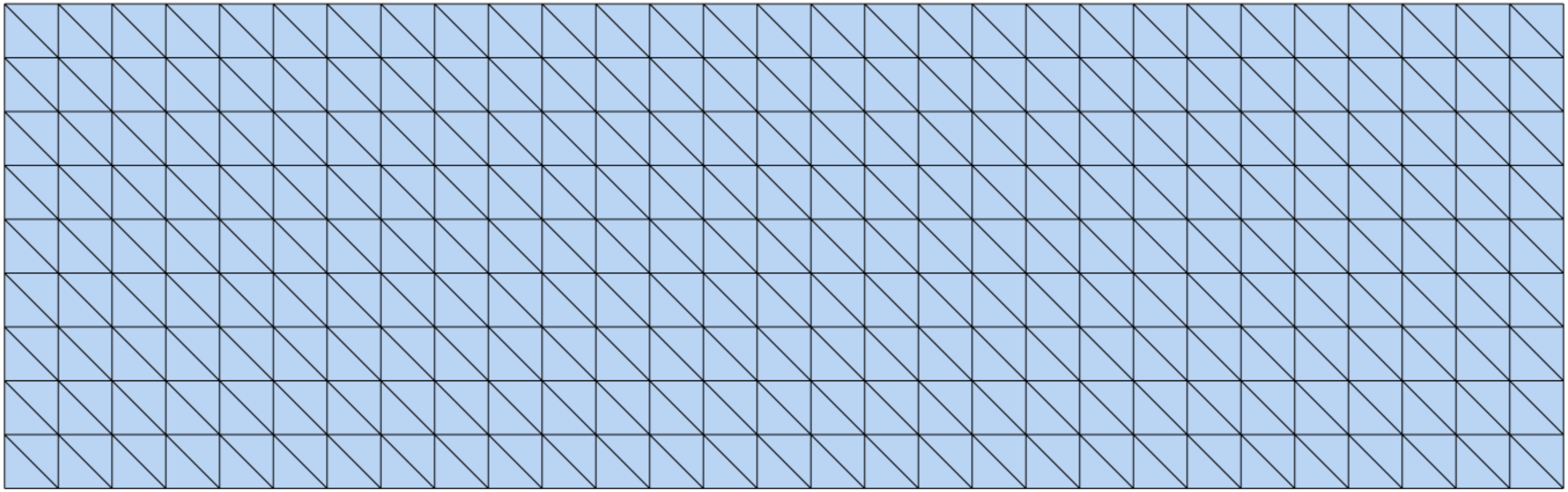

- $p$ mapped to average of neighborhood!
- By construction, the discrete case



# Tutte’s embedding



# Computing maps



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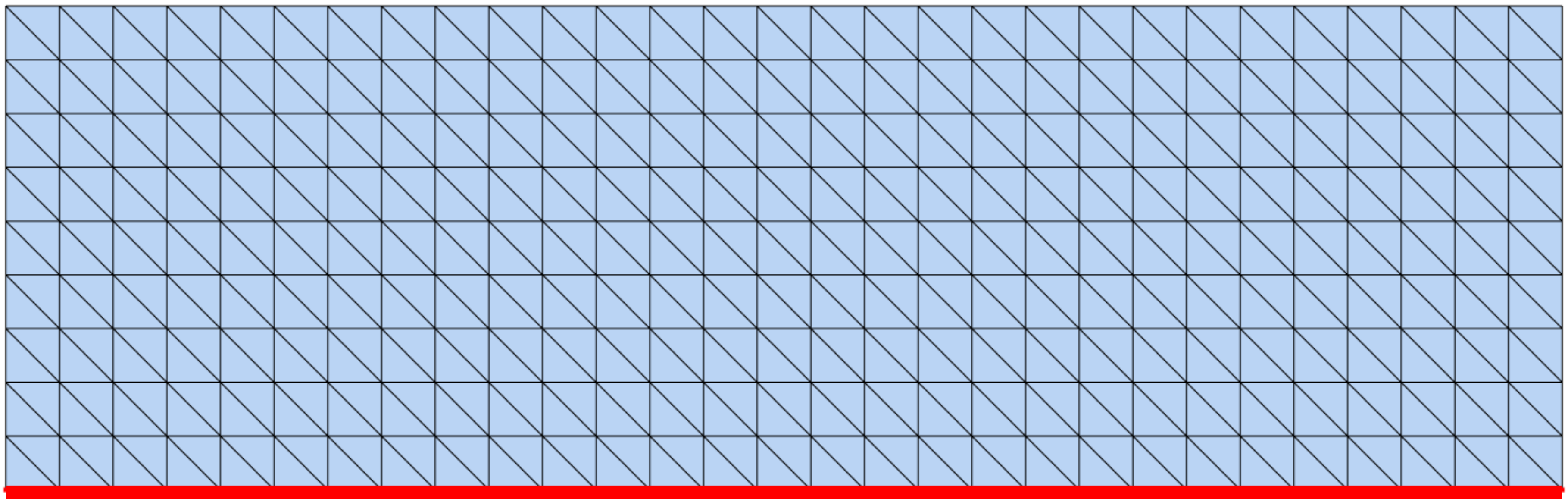

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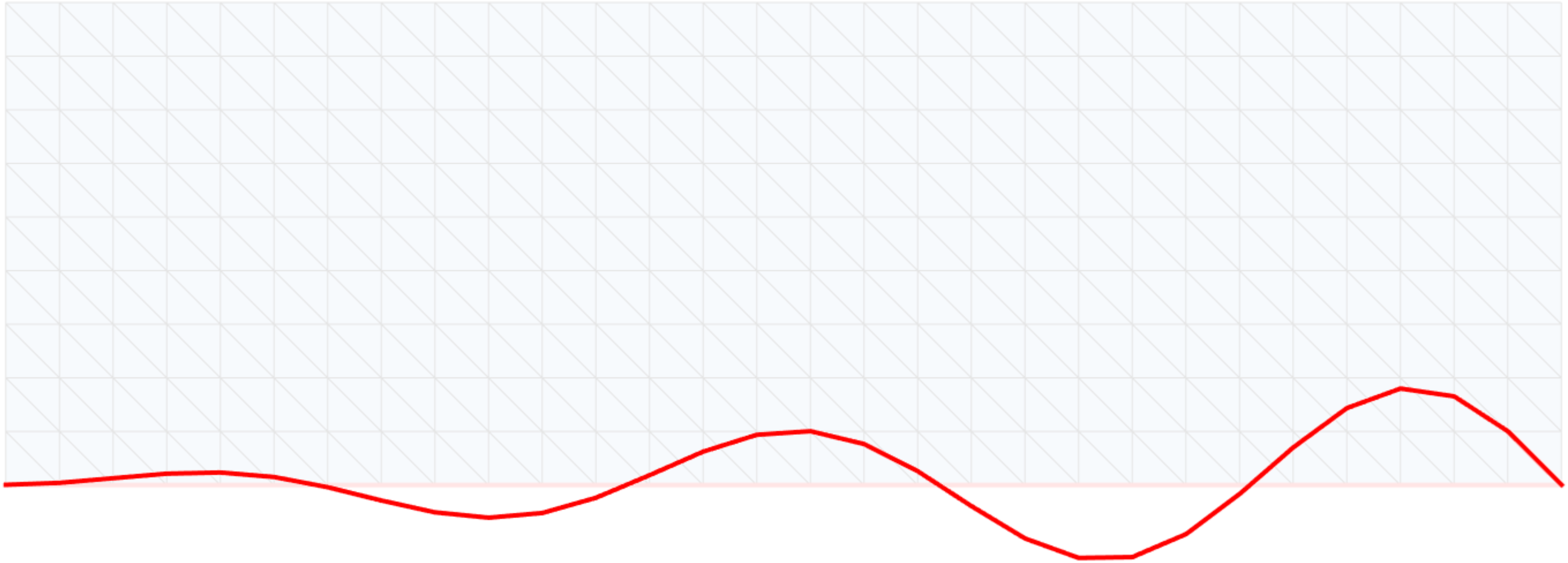

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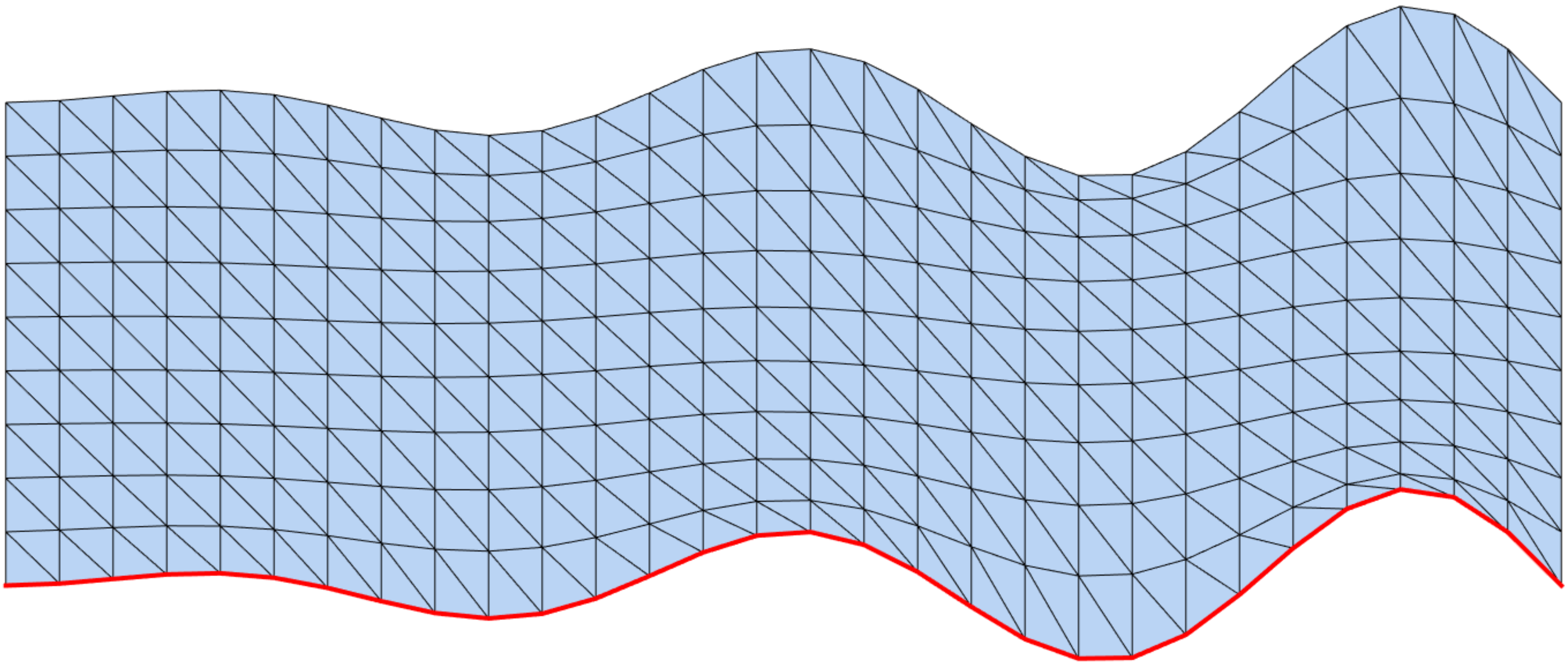
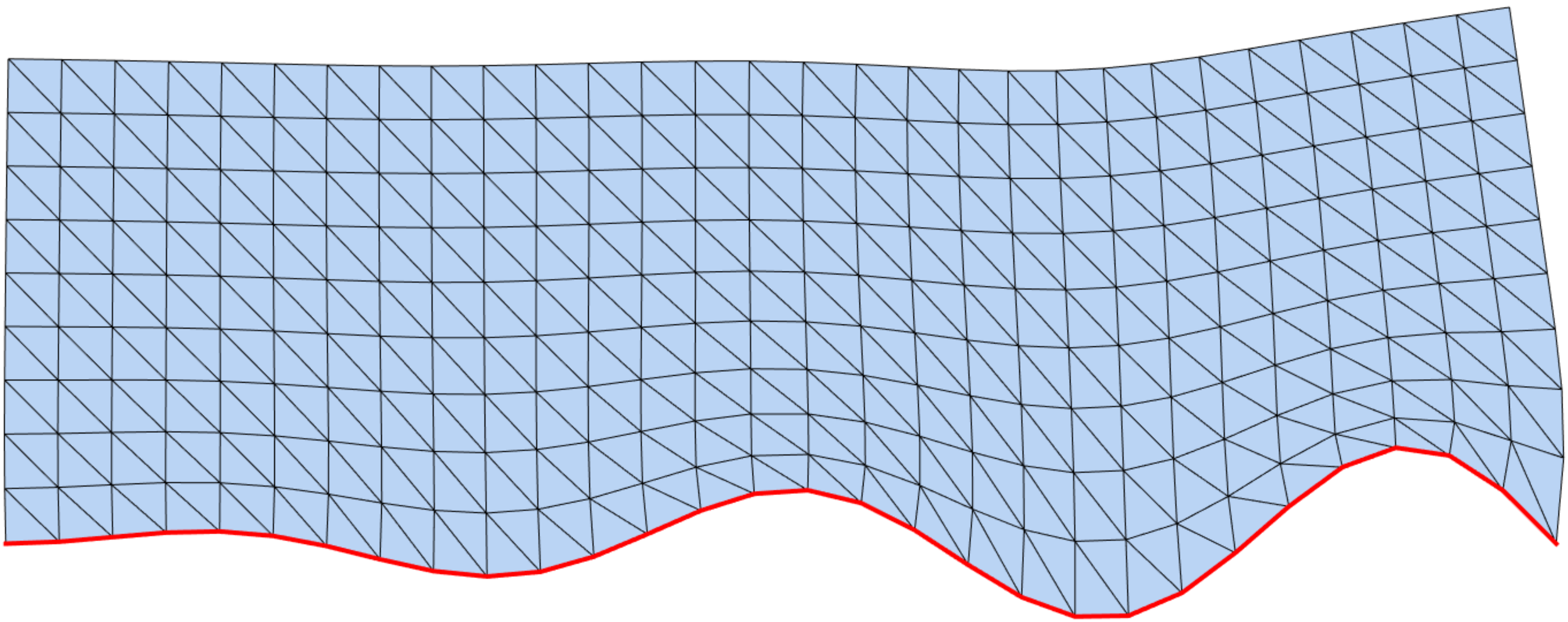
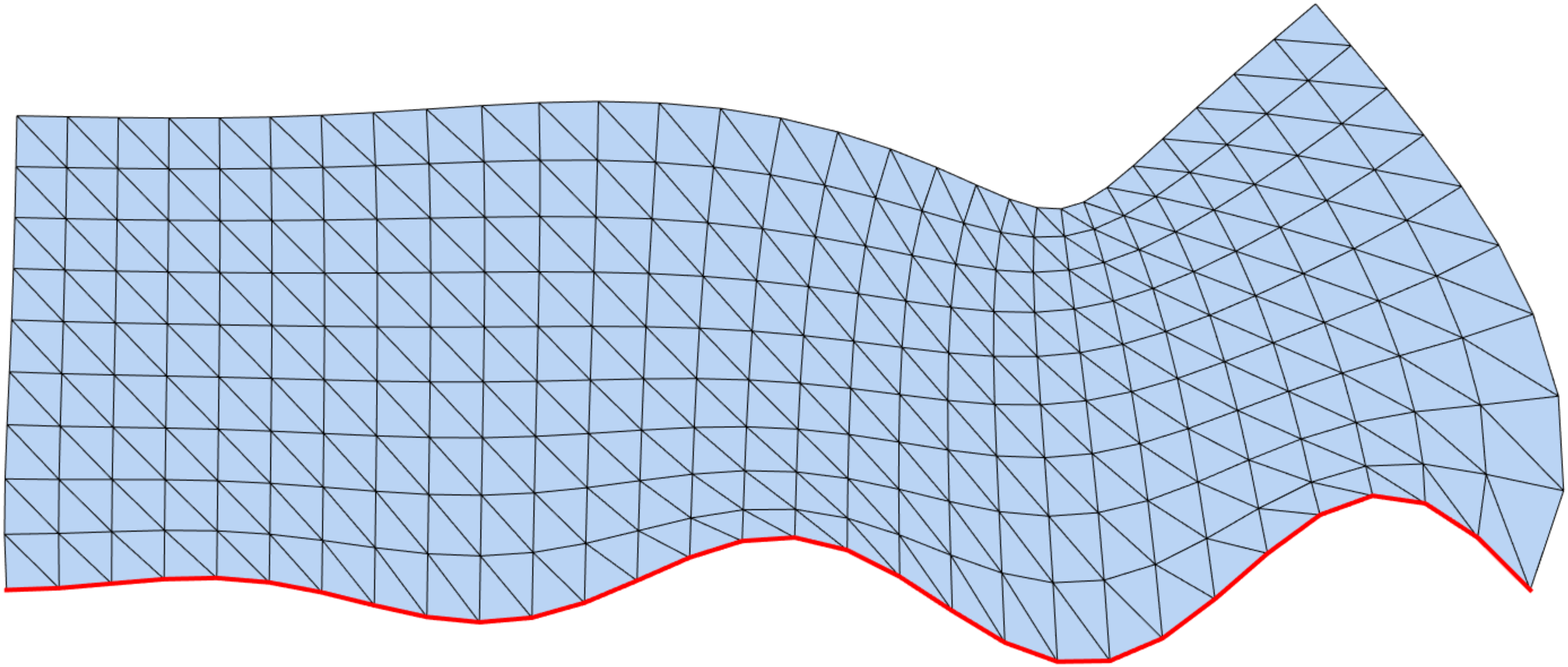


# Computing maps

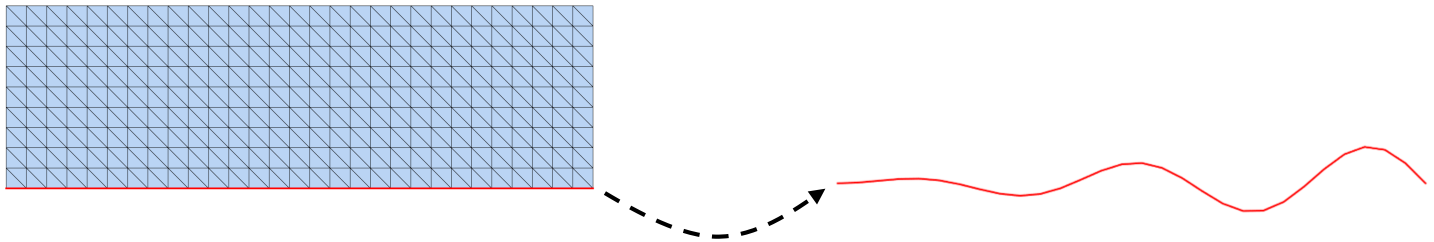


# Computing maps

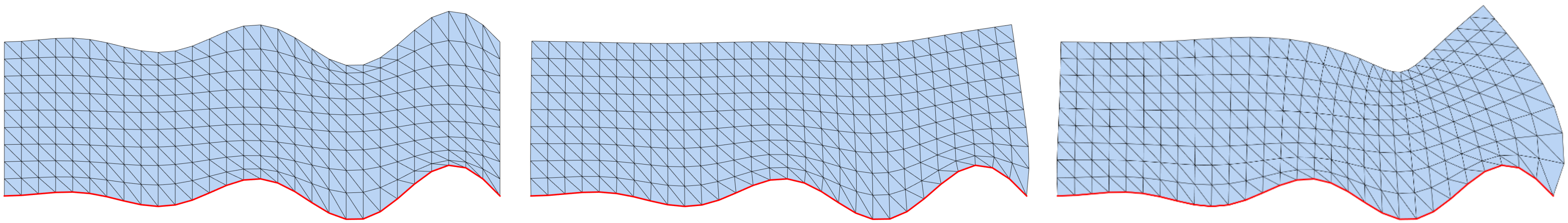


# Computing maps

- Imposing constraints



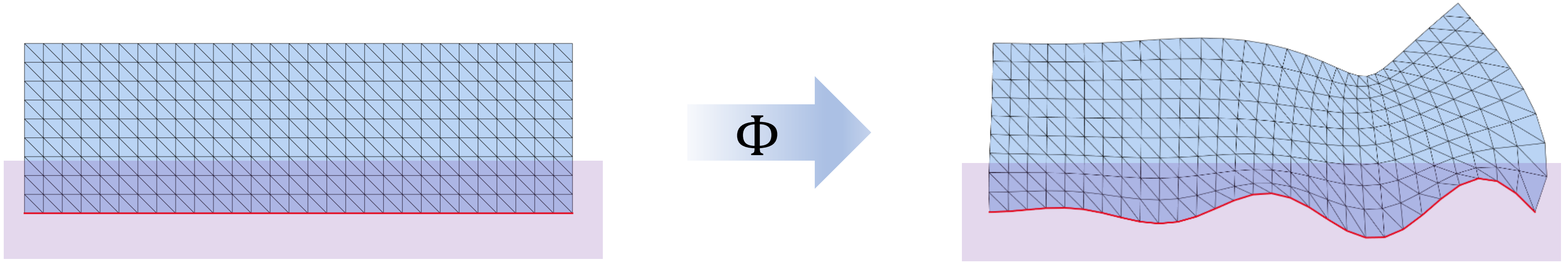
- Finding maps that are most…



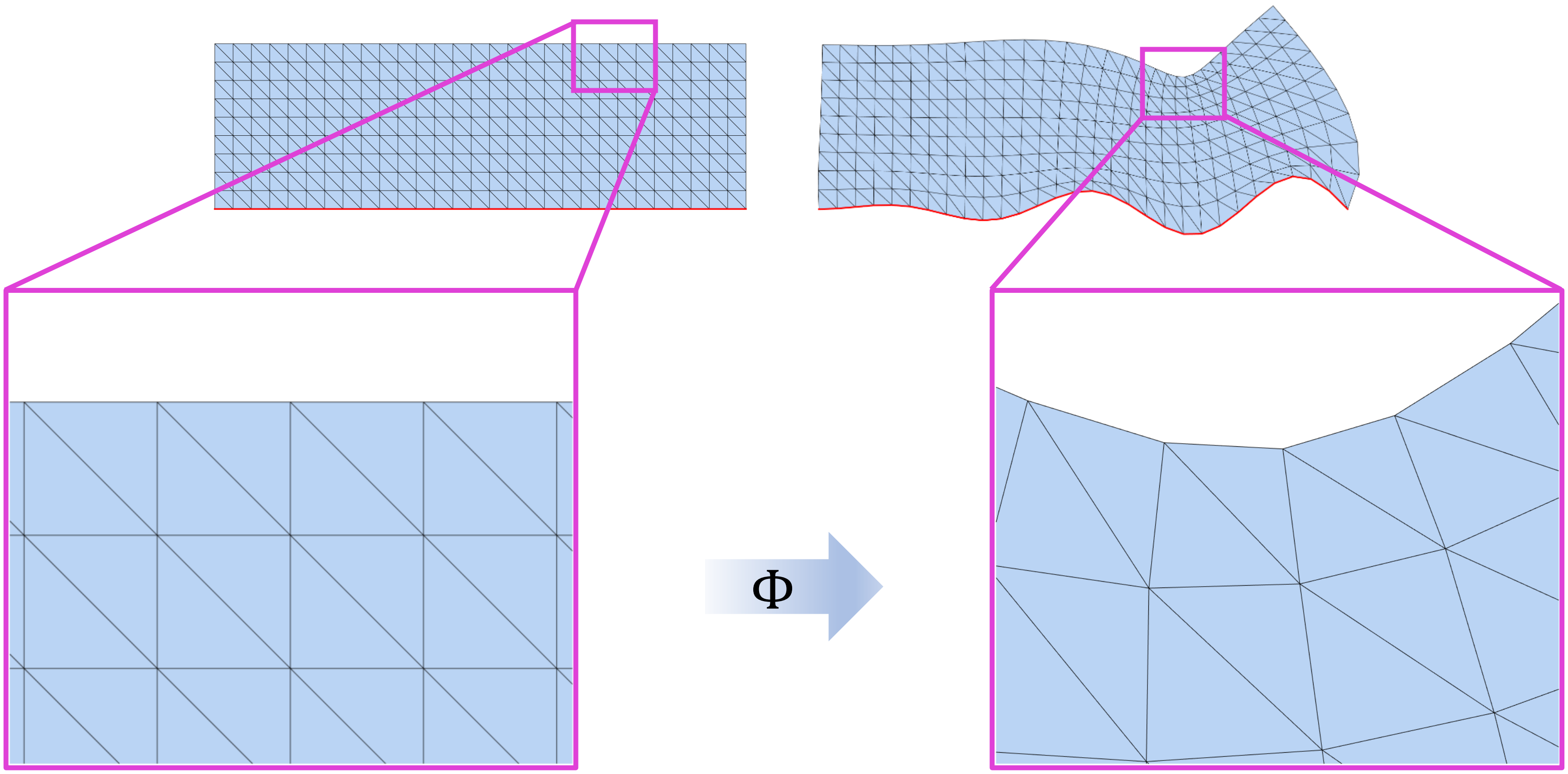
# Constrained Optimization

$$\underset{\Phi}{\operatorname{argmin}}\; E(\Phi) \qquad \text{Energy}$$

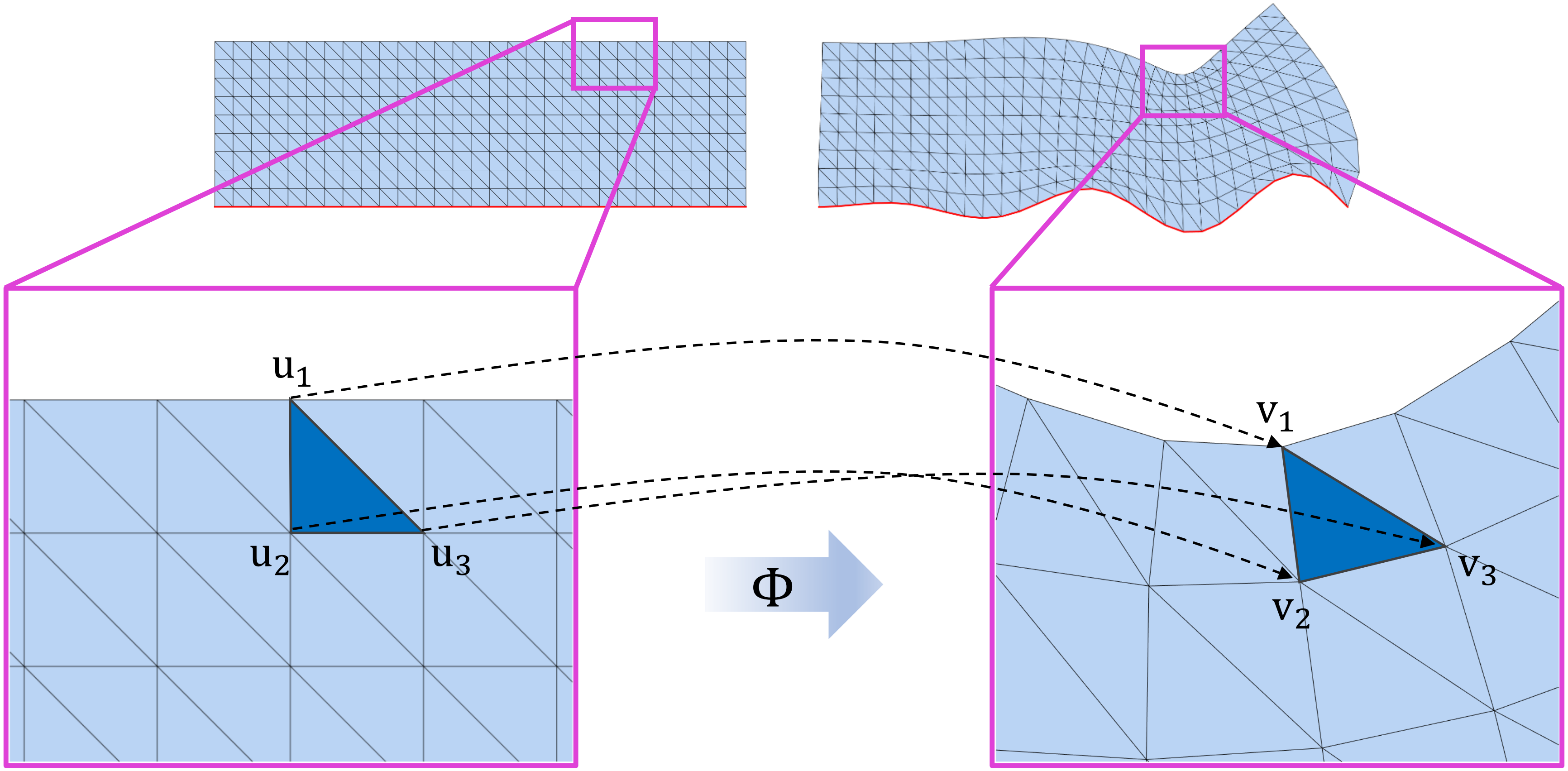
$$\text{s.t.}\quad \Phi \in K \qquad \text{Constraints}$$



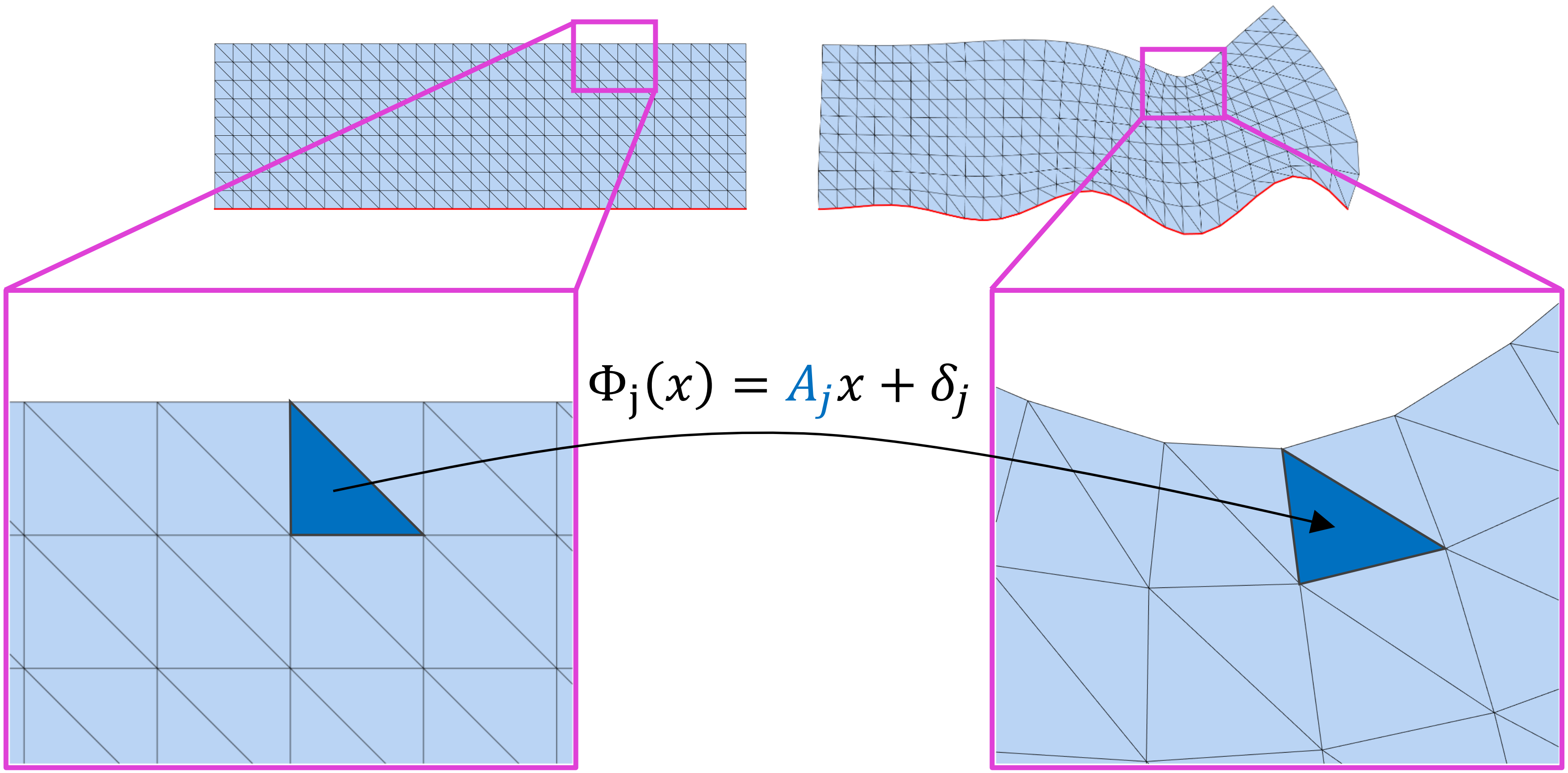
# Energy



# Energy



# Energy

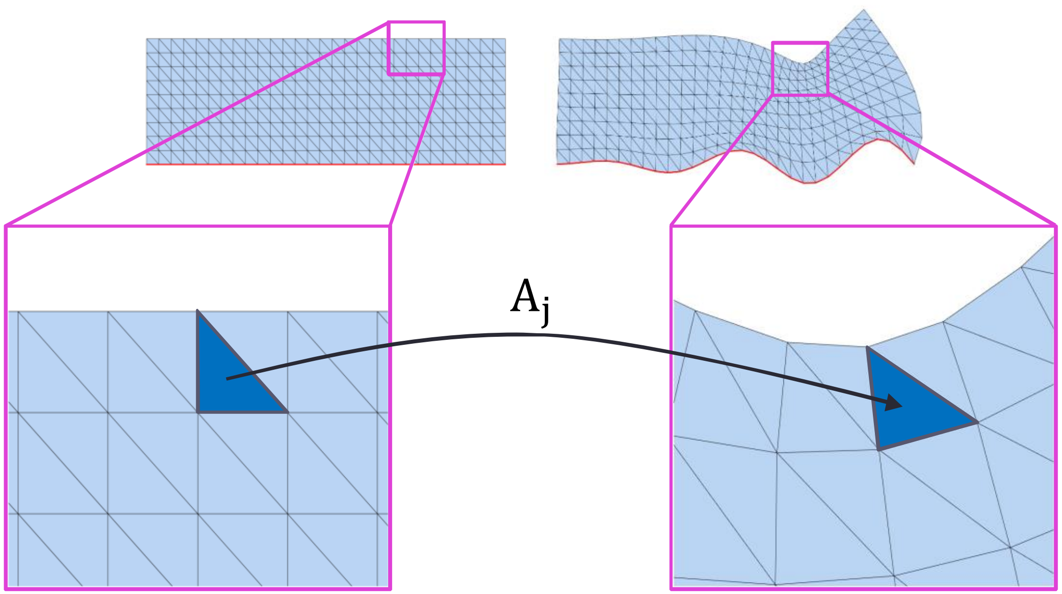



$$E(\Phi) = E(A_1, \ldots, A_m)$$

# Map optimization

- In terms of differentials:

$$\operatorname{argmin} E(\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{m})$$

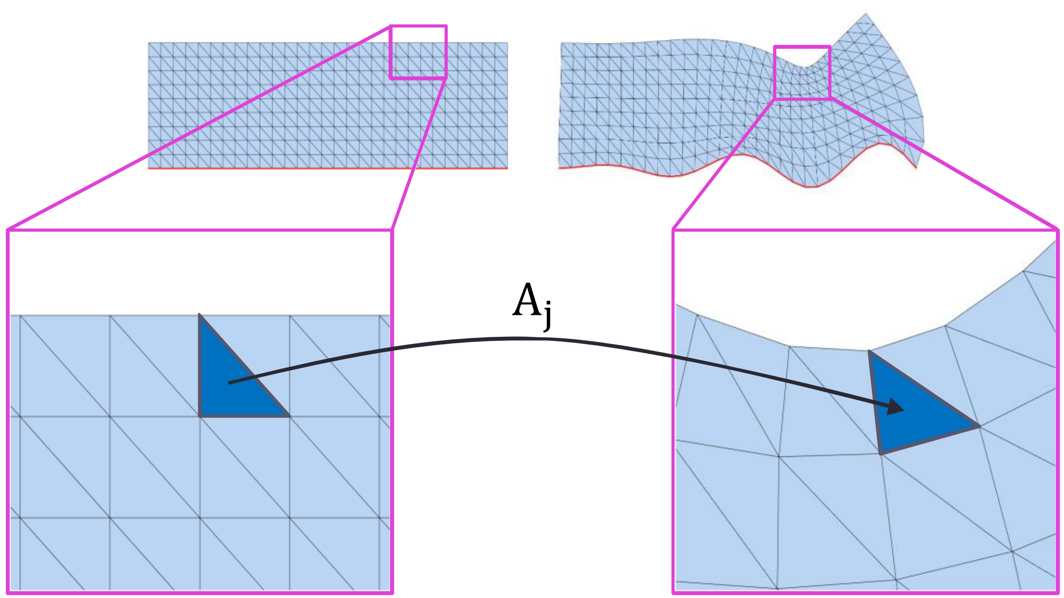


# Map optimization

- In terms of differentials:

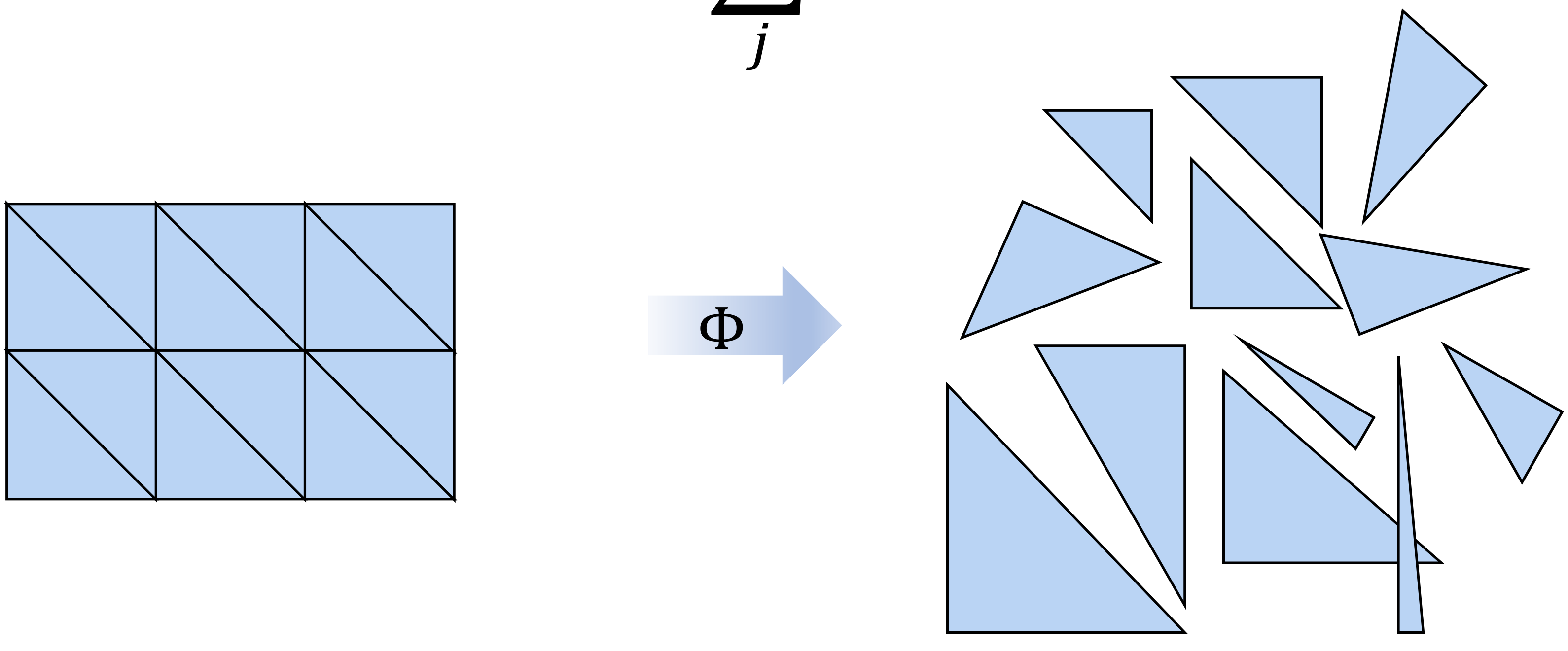
$$\operatorname{argmin} \sum_j f(A_j)$$

Separable



# Map optimization

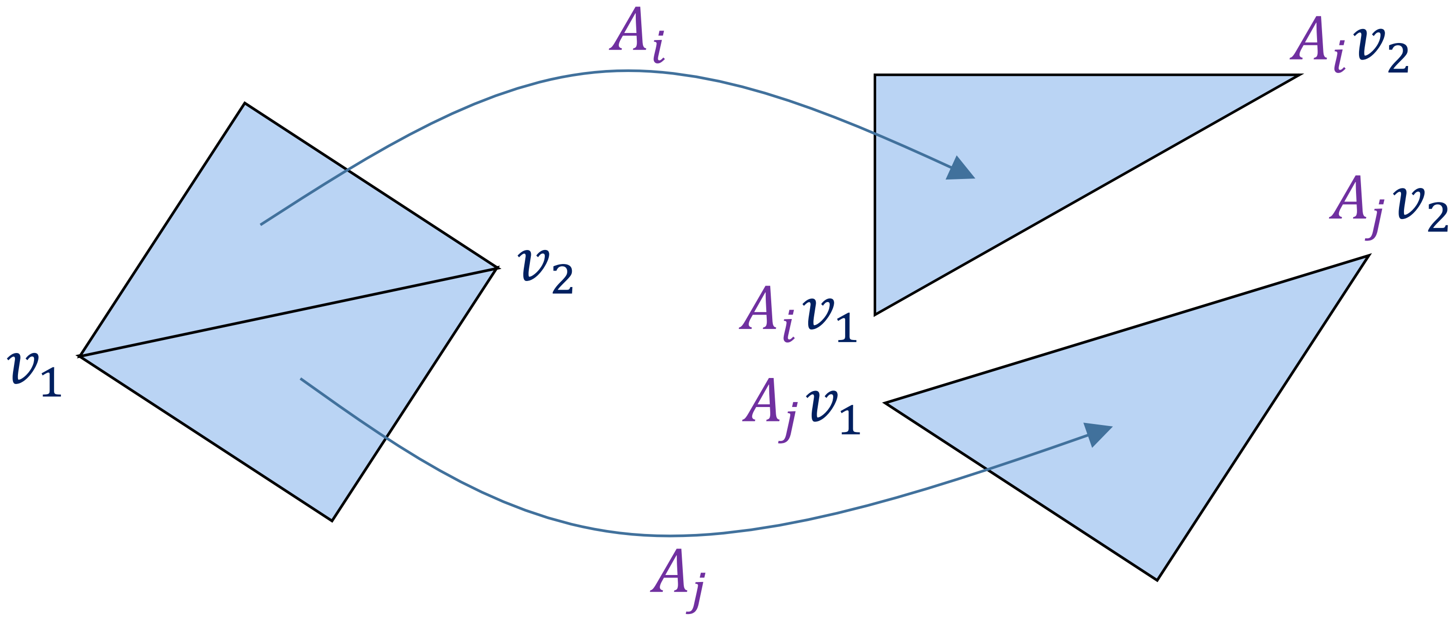
$$\operatorname{argmin} \sum_j f(A_j)$$




Must impose continuity!

# Explicit continuity

- Optimization variables: $A_1, A_2, \ldots, A_m$
- Adjacent $A_j$'s must agree



# Explicit continuity

- Optimization variables: $A_1, A_2, \ldots, A_m$
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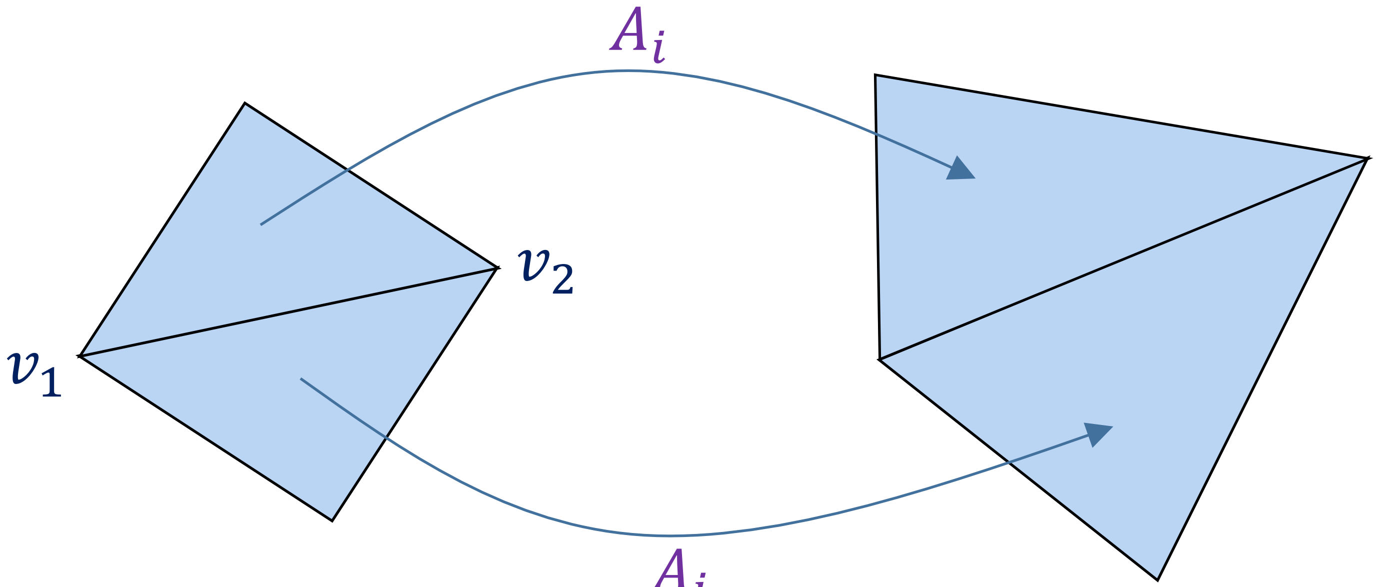
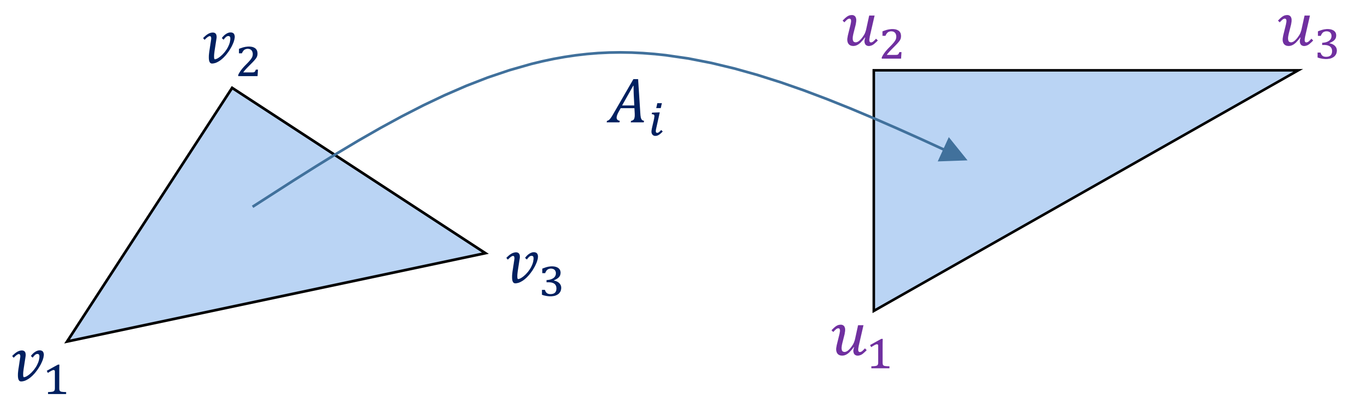



$$A_i v_1 = A_j v_1$$
$$A_i v_2 = A_j v_2$$

# Implicit continuity



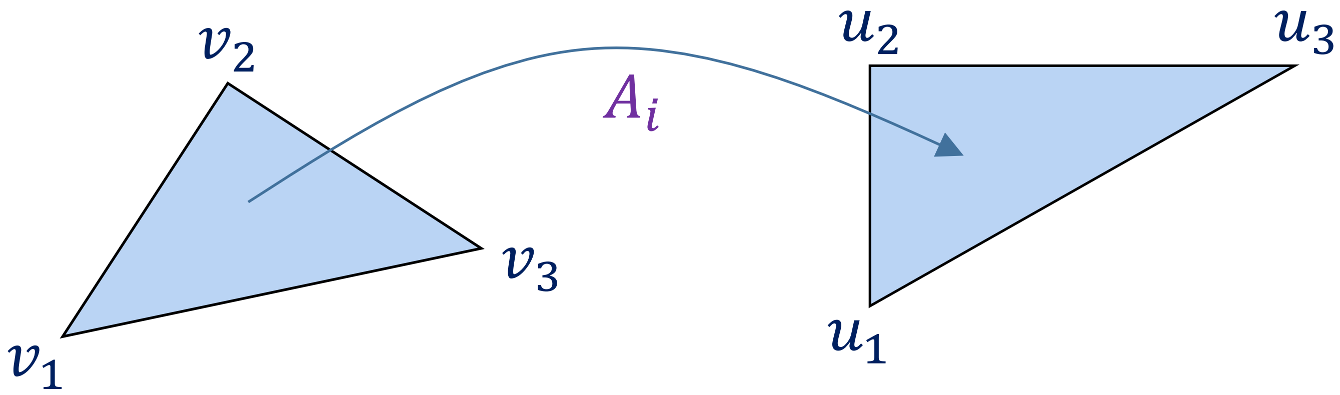

$$A_i\overline{[v_1\ v_2\ v_3]} = \overline{[u_1\ u_2\ u_3]}$$

$$A_i = \overline{[u_1\ u_2\ u_3]}\,\overline{[v_1\ v_2\ v_3]}^{\dagger}$$

$$A_i = A_i(U)$$

Linearly express $A_i$'s in terms of $U$

# Implicit continuity

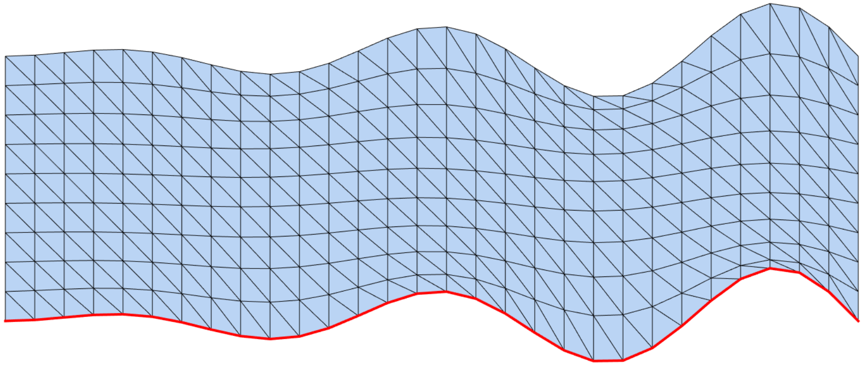


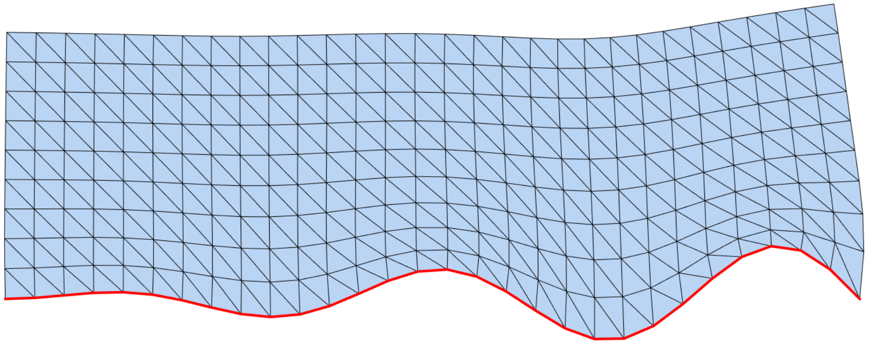

- Optimization variables: $u_1, u_2, \dots, u_n \quad (U)$

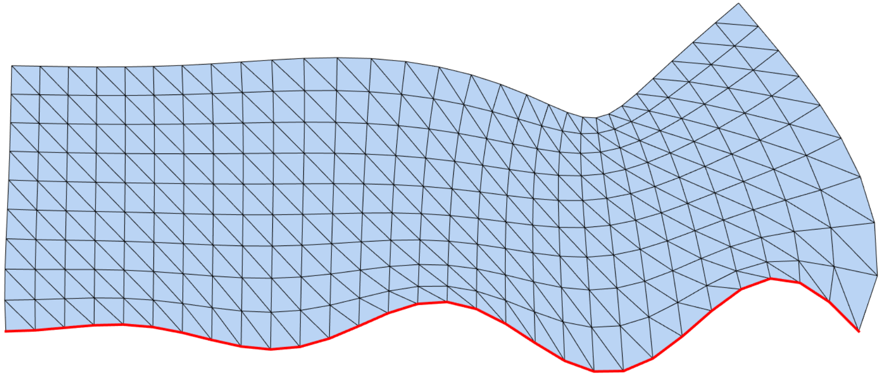
$$E(\Phi) = \sum_j f\left(A_j(U)\right)$$

# Popular energies

$$\operatorname{argmin} \sum_j f(A_j)$$



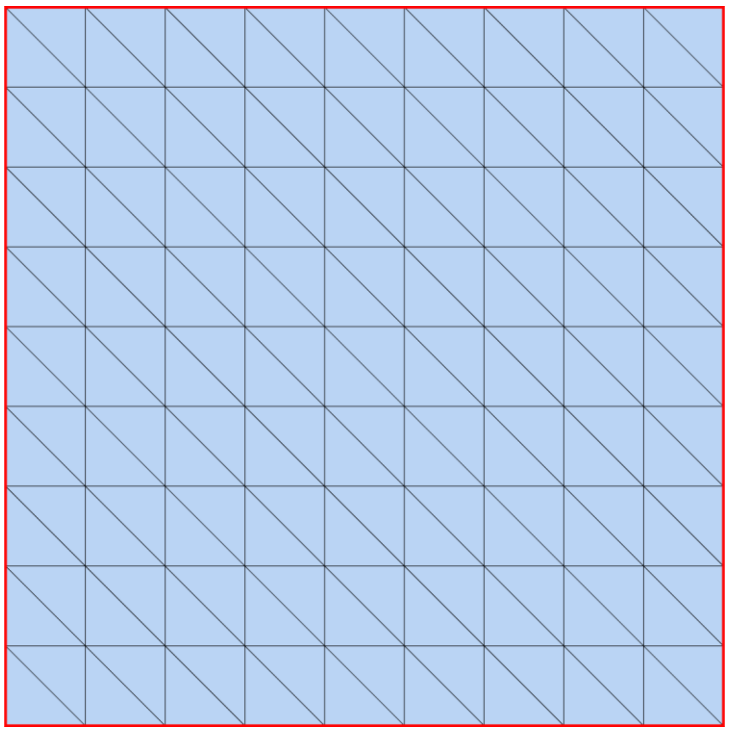




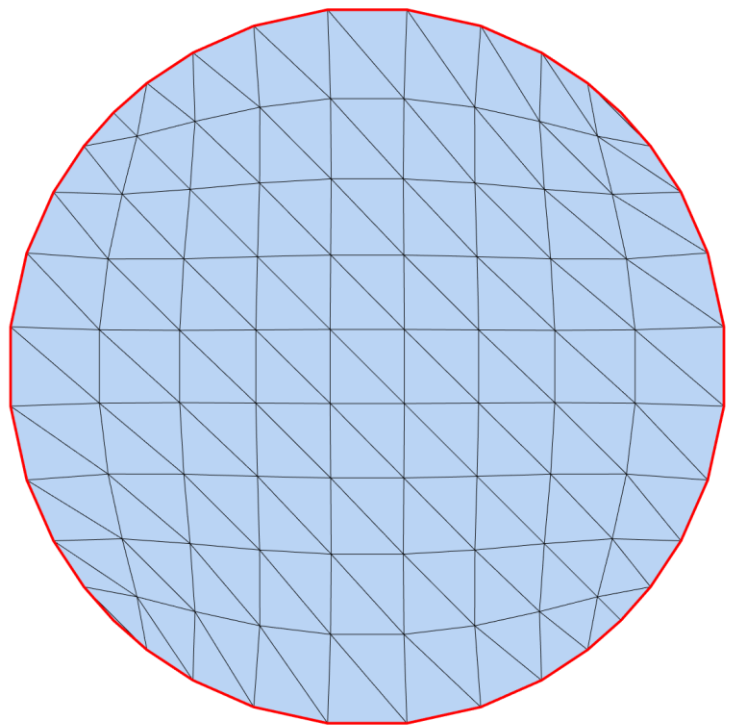
# Dirichlet

area / volume

$$E_D = \sum_j w_j \left\| A_j \right\|_F^2$$

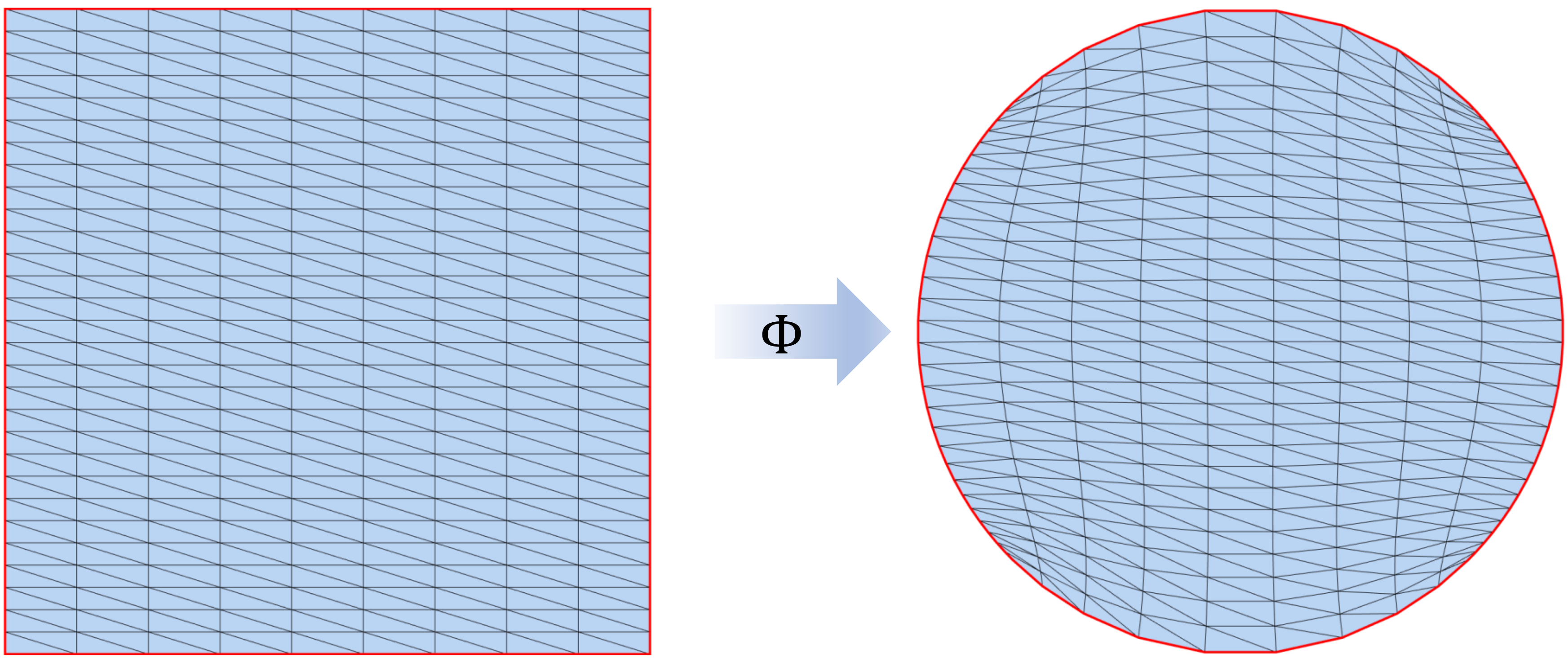


$\Phi$

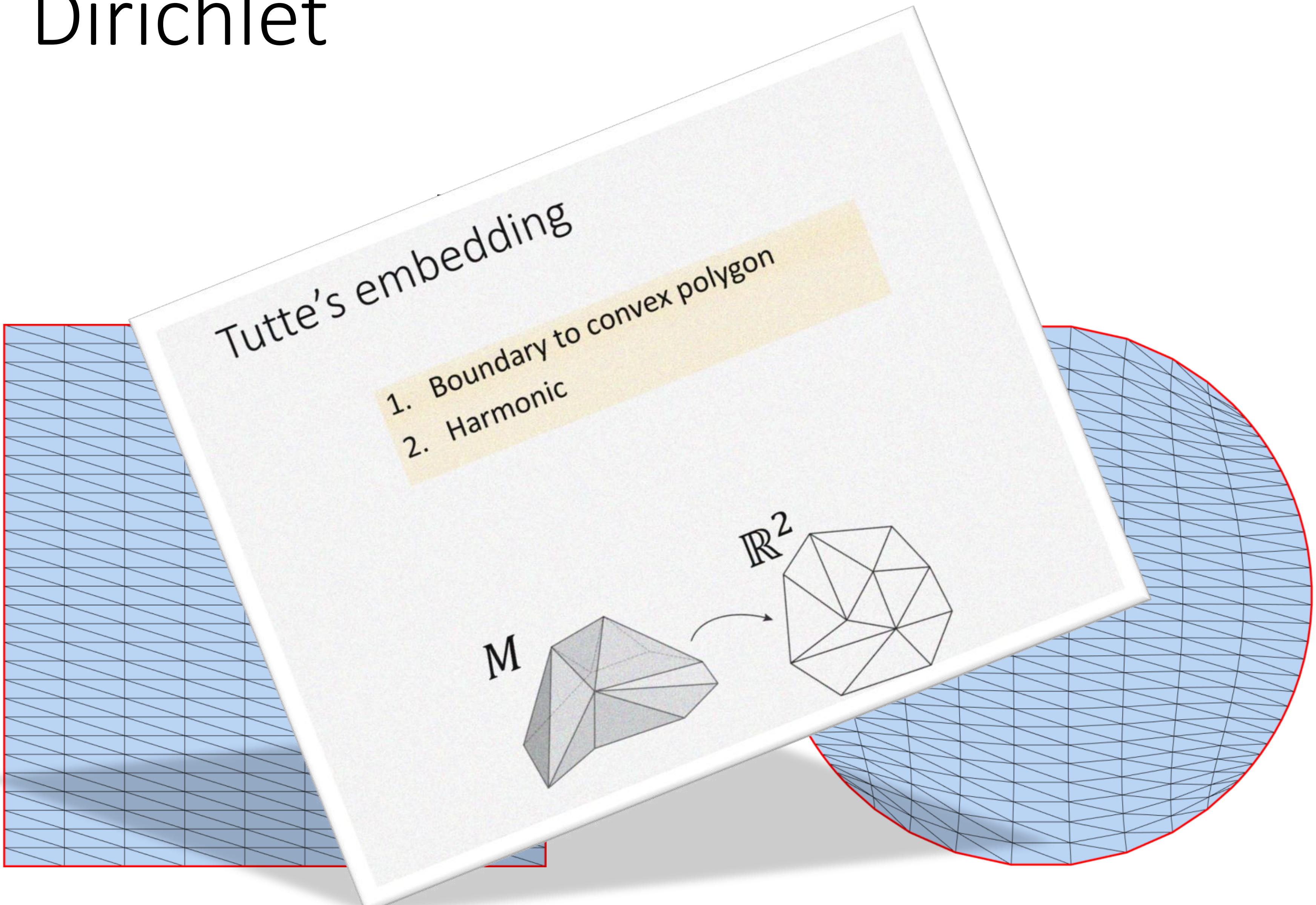


# Dirichlet

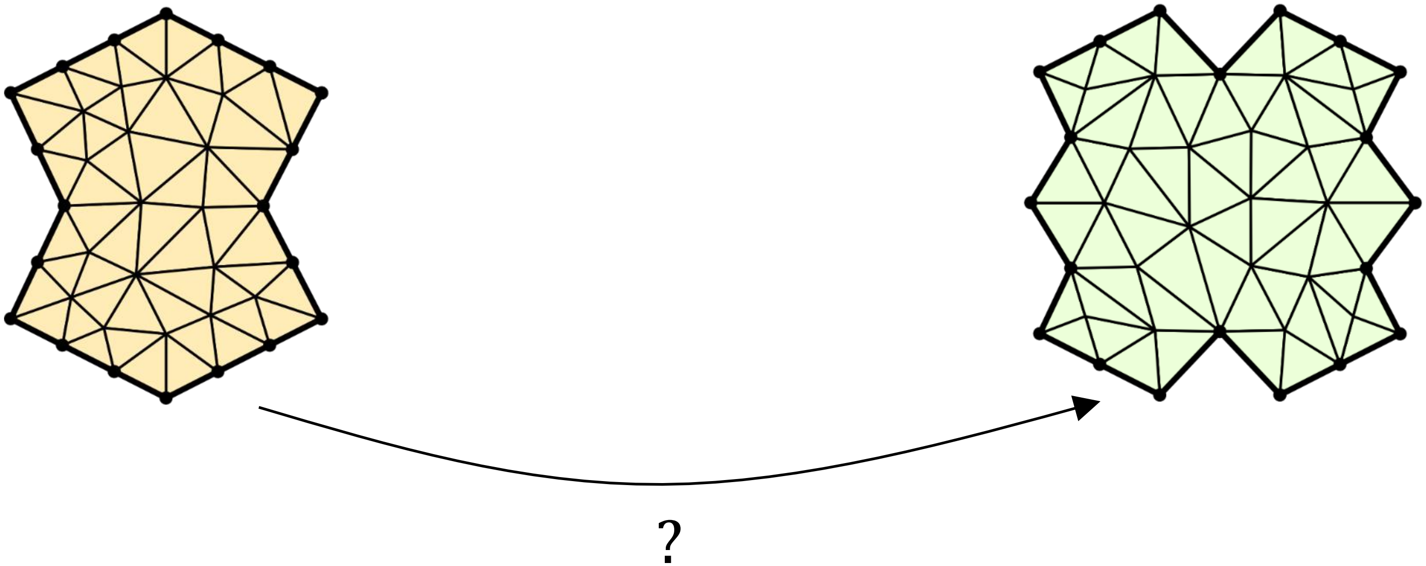
$$E_D = \sum_j w_j \|A_j\|_F^2$$



# Dirichlet

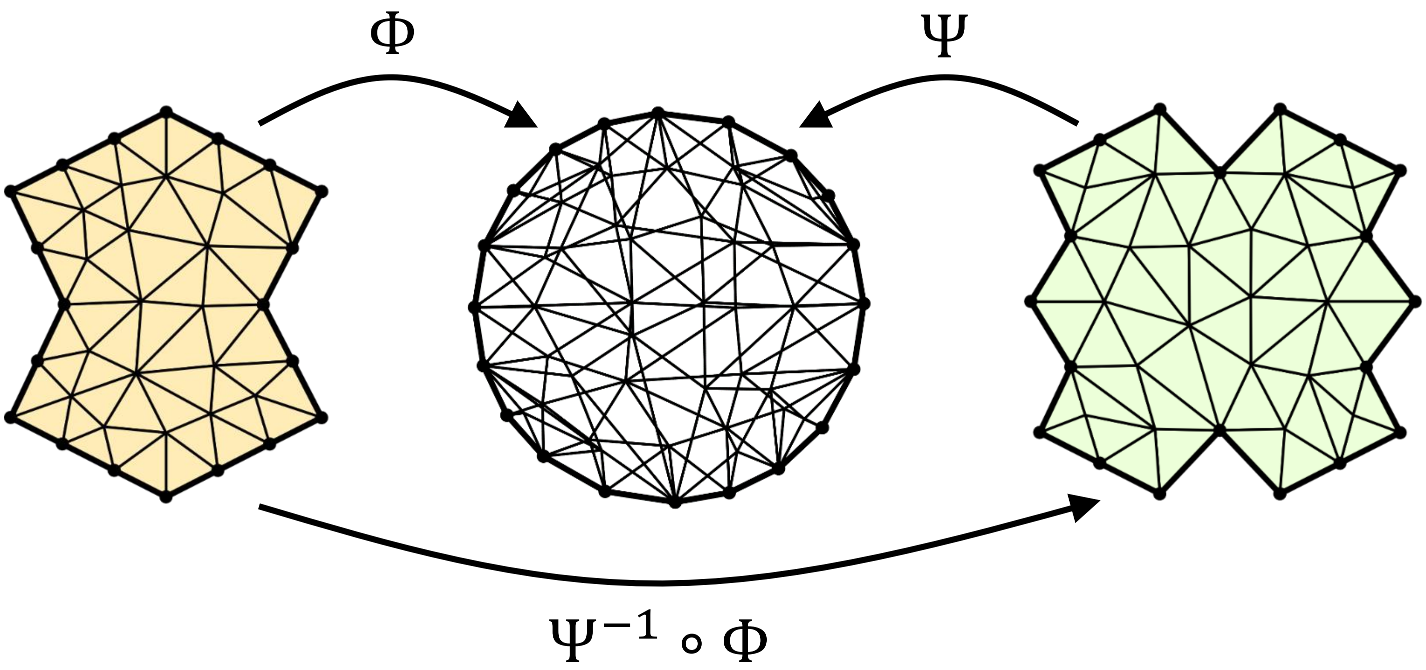

Tutte's embedding
1. Boundary to convex polygon
2. Harmonic
M
$\mathbb{R}^2$

# Dirichlet



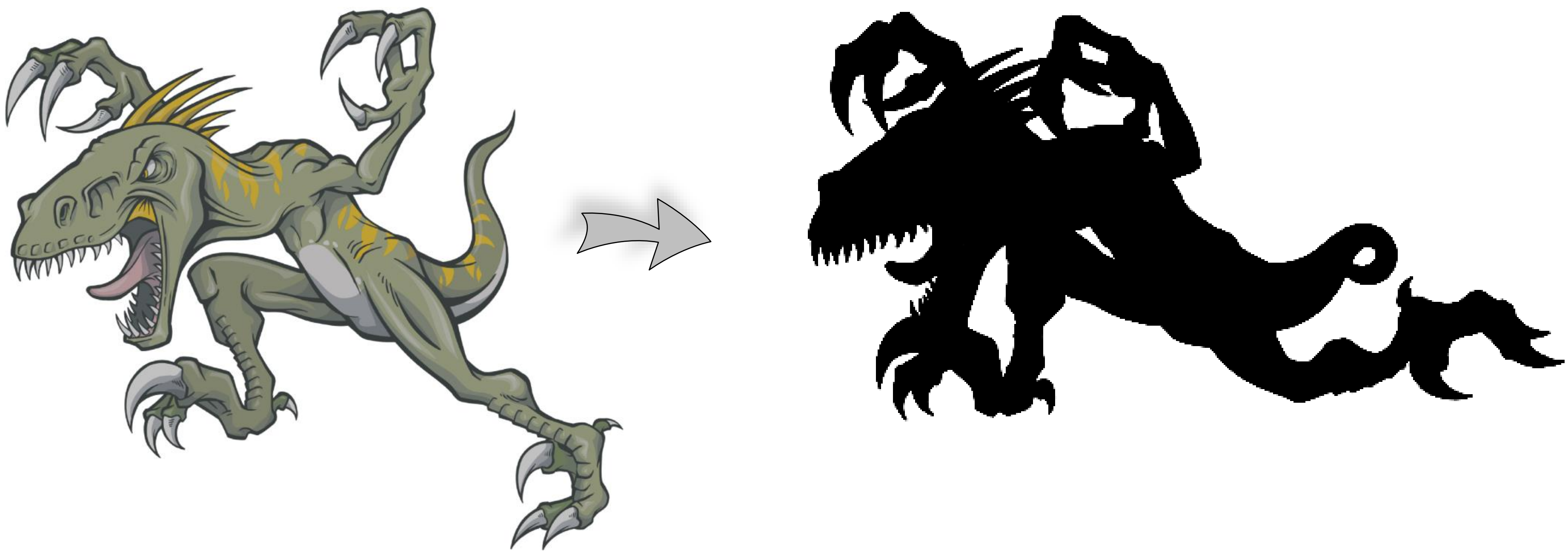
[Weber & Zorin 2014]

# Dirichlet




[Weber & Zorin 2014]

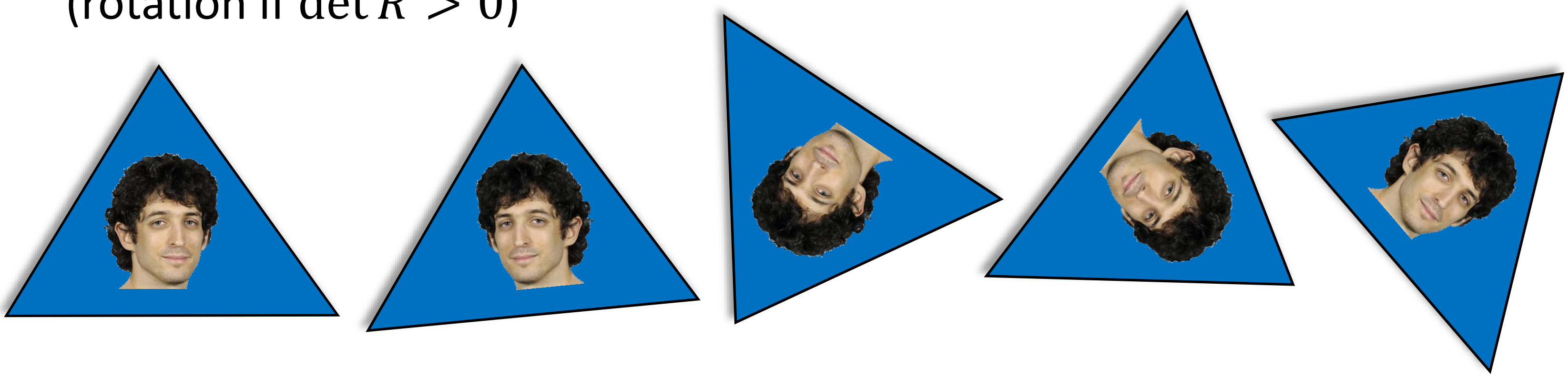
# Dirichlet



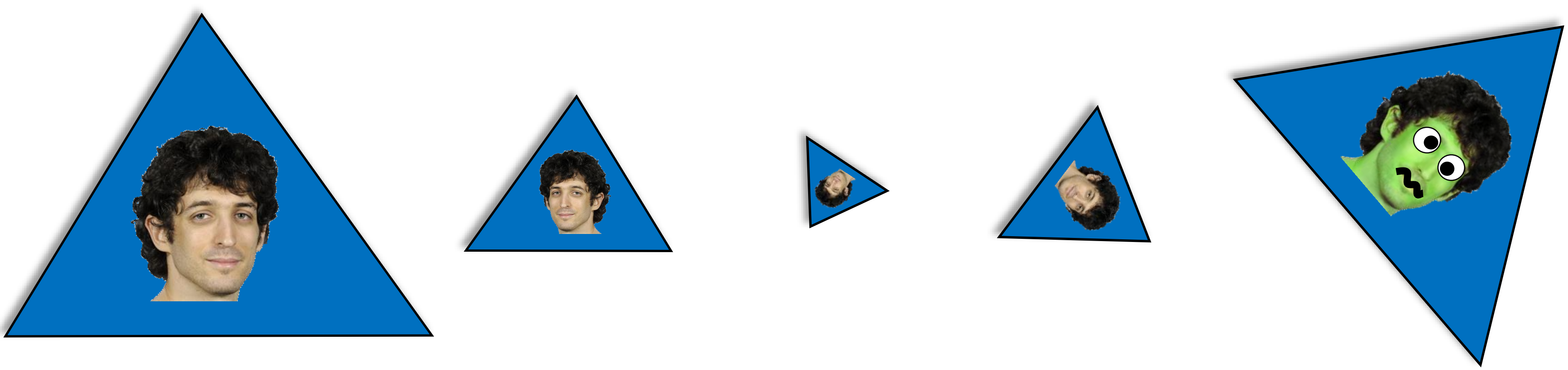
[Weber & Zorin 2014]

# Orthogonal and Similarity

- $R$ is <u>orthogonal</u> if $R^T = R^{-1}$
  (rotation if $\det R > 0$)



- $S$ is a <u>similarity</u> if $S = \alpha R$



# Closest rotation/similarity

- $\mathcal{R}(A)$ = closest orthogonal/rotation matrix to $A$
- $\mathcal{S}(A)$ = closest similarity matrix to $A$

- Computable using **SVD**:

$$A = U\Sigma V^T; \qquad \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

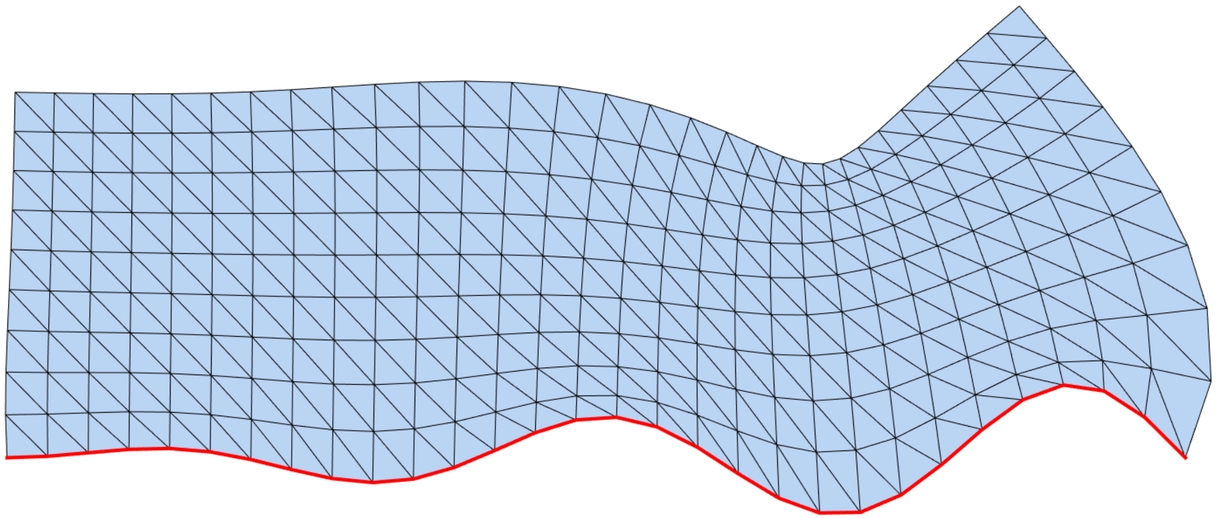
- $\mathcal{R}(A) = U\cancel{\Sigma}V^T = UV^T$
- $\mathcal{S}(A) = \overline{\sigma}UV^T$

mean of SVs

# Least Squares Conformal Map (LSCM)

$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2$$

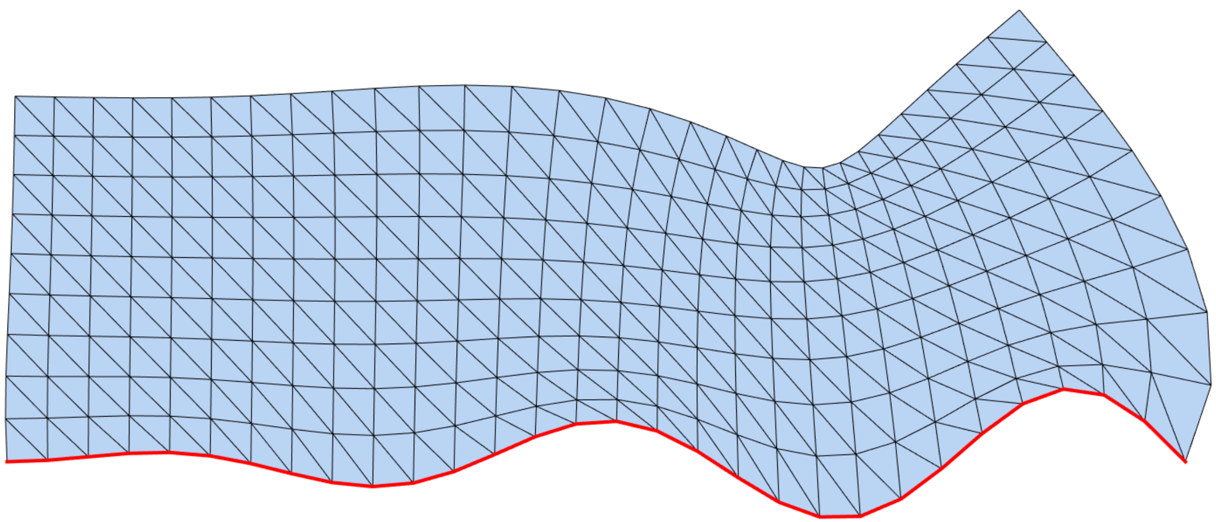
closest similarity



# Least Squares Conformal Map (LSCM)

$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2$$

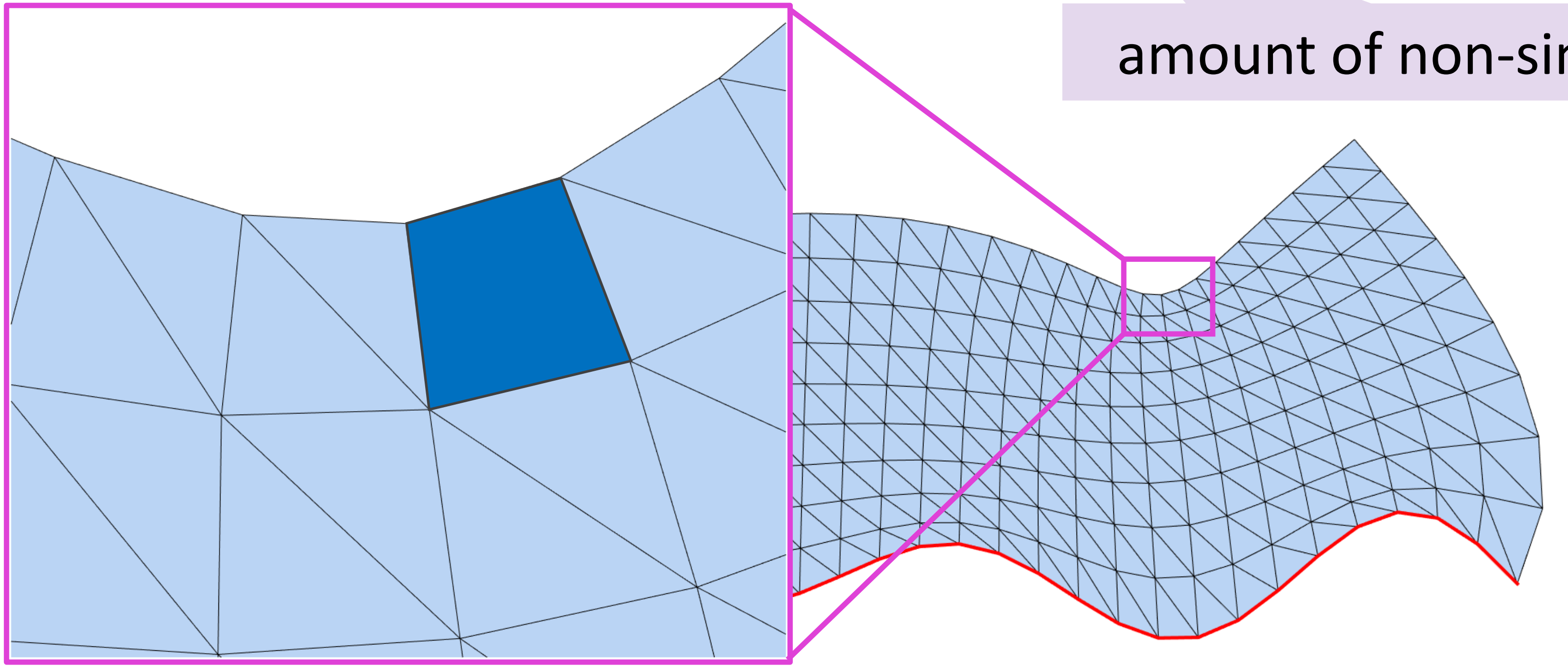
amount of non-similarity



# Least Squares Conformal Map (LSCM)

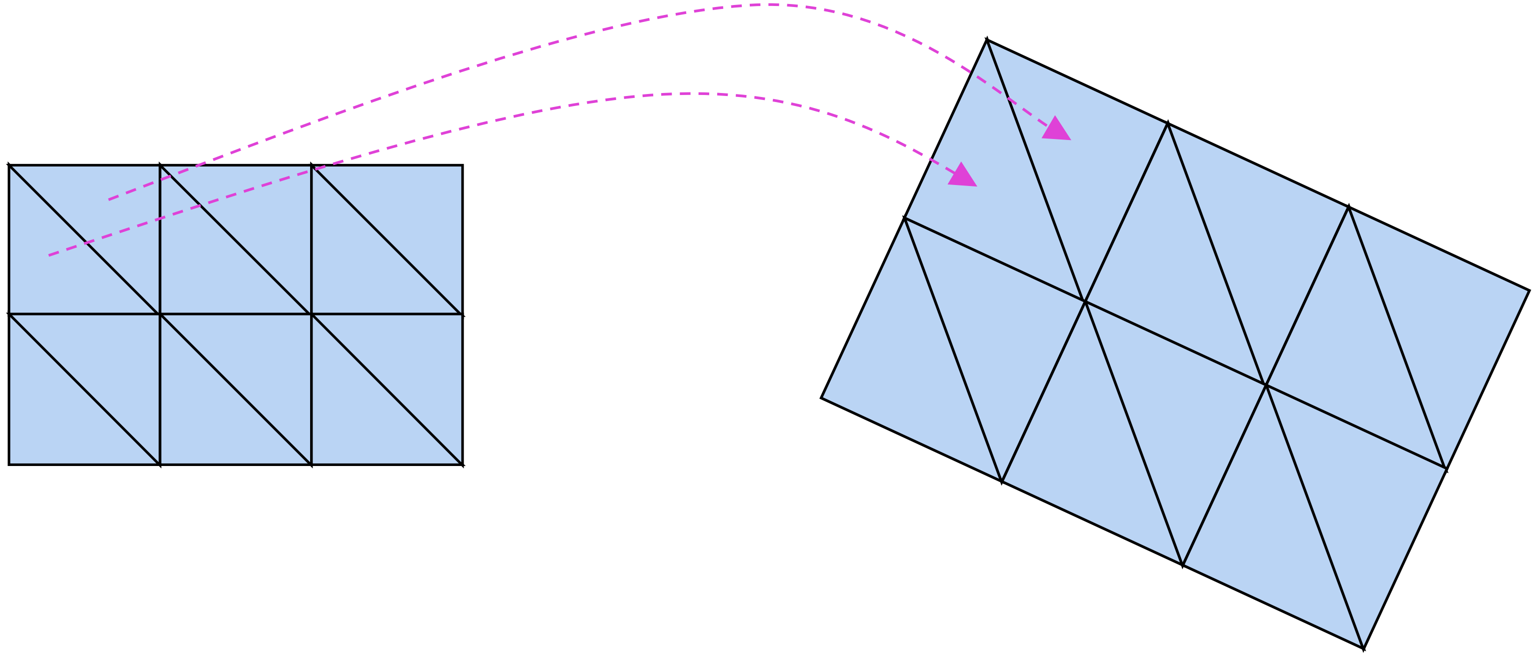
$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2$$

amount of non-similarity



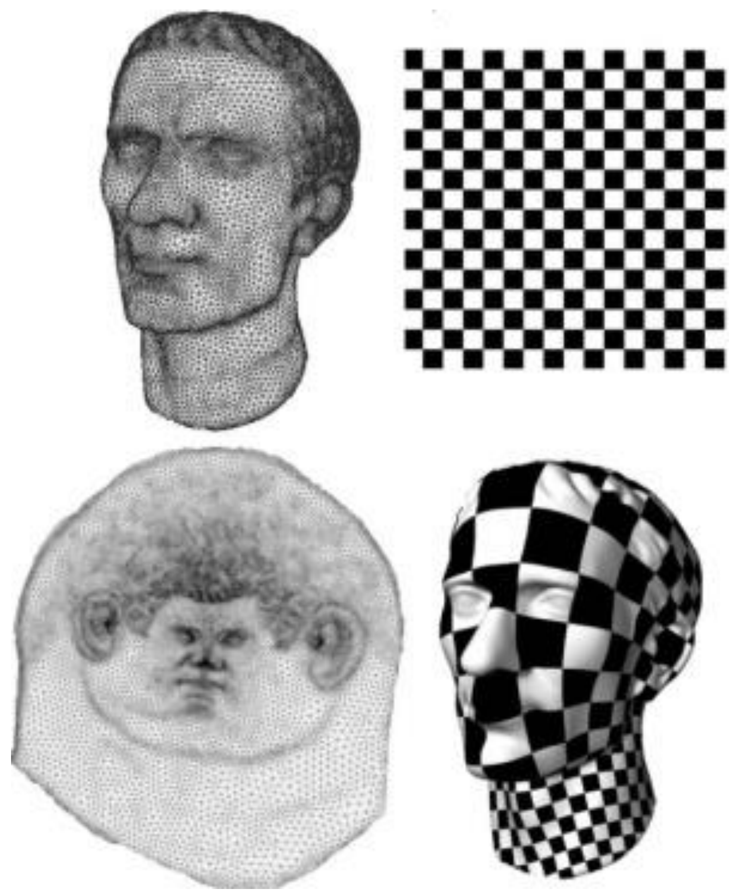
# Least Squares Conformal Map (LSCM)

$$E_L = \sum_j w_j \left\| A_j - \mathcal{S}(A_j) \right\|_F^2 = 0$$



global similarity = discrete conformal maps

# Least Squares Conformal Map (LSCM)

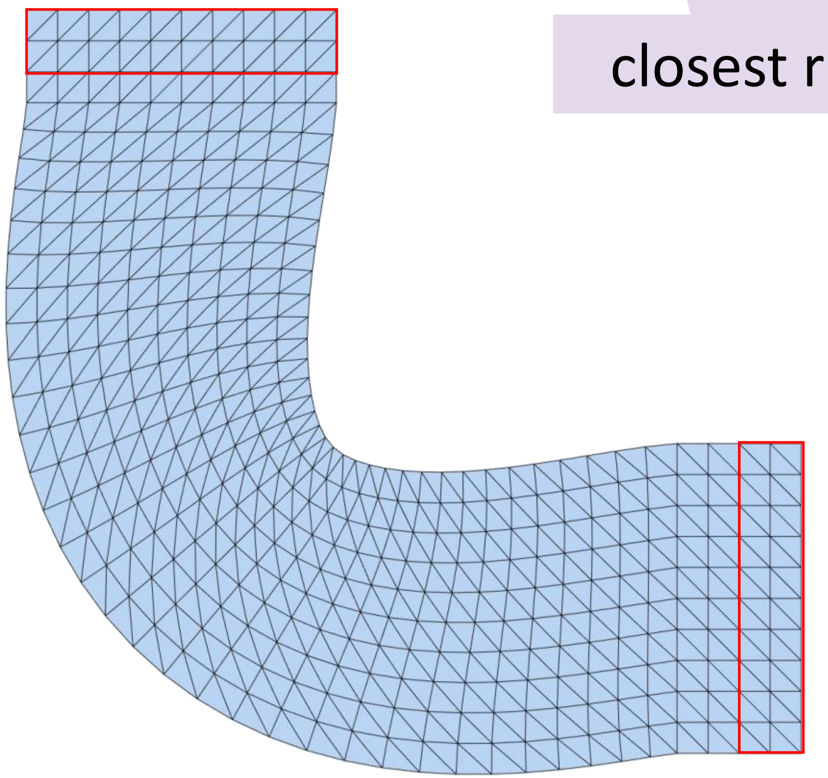


[Lévy et al. 2002]

# As-Rigid-As-Possible (ARAP)

$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

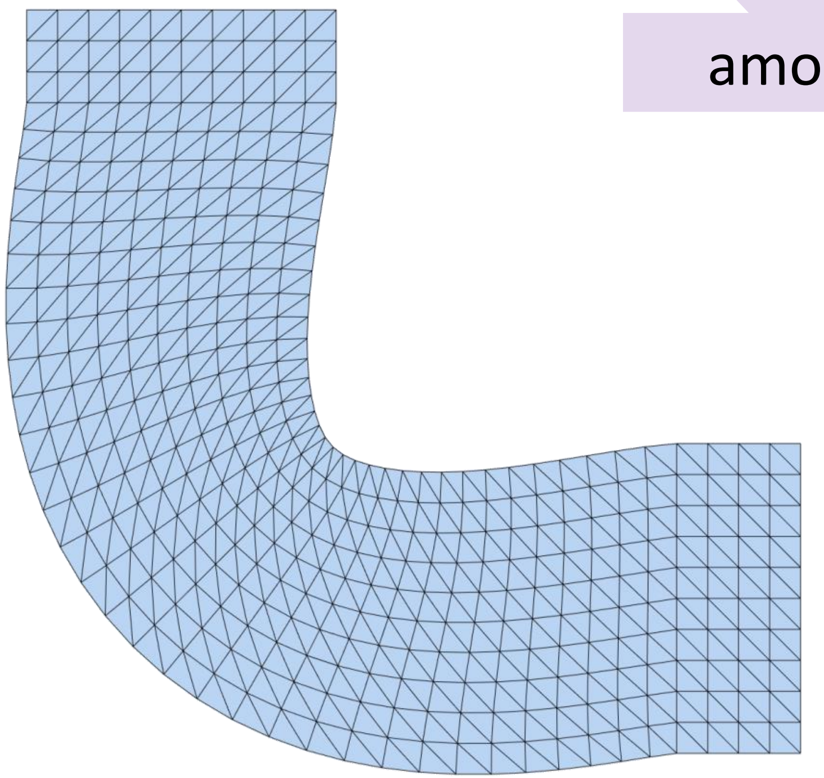
closest rigid transformation



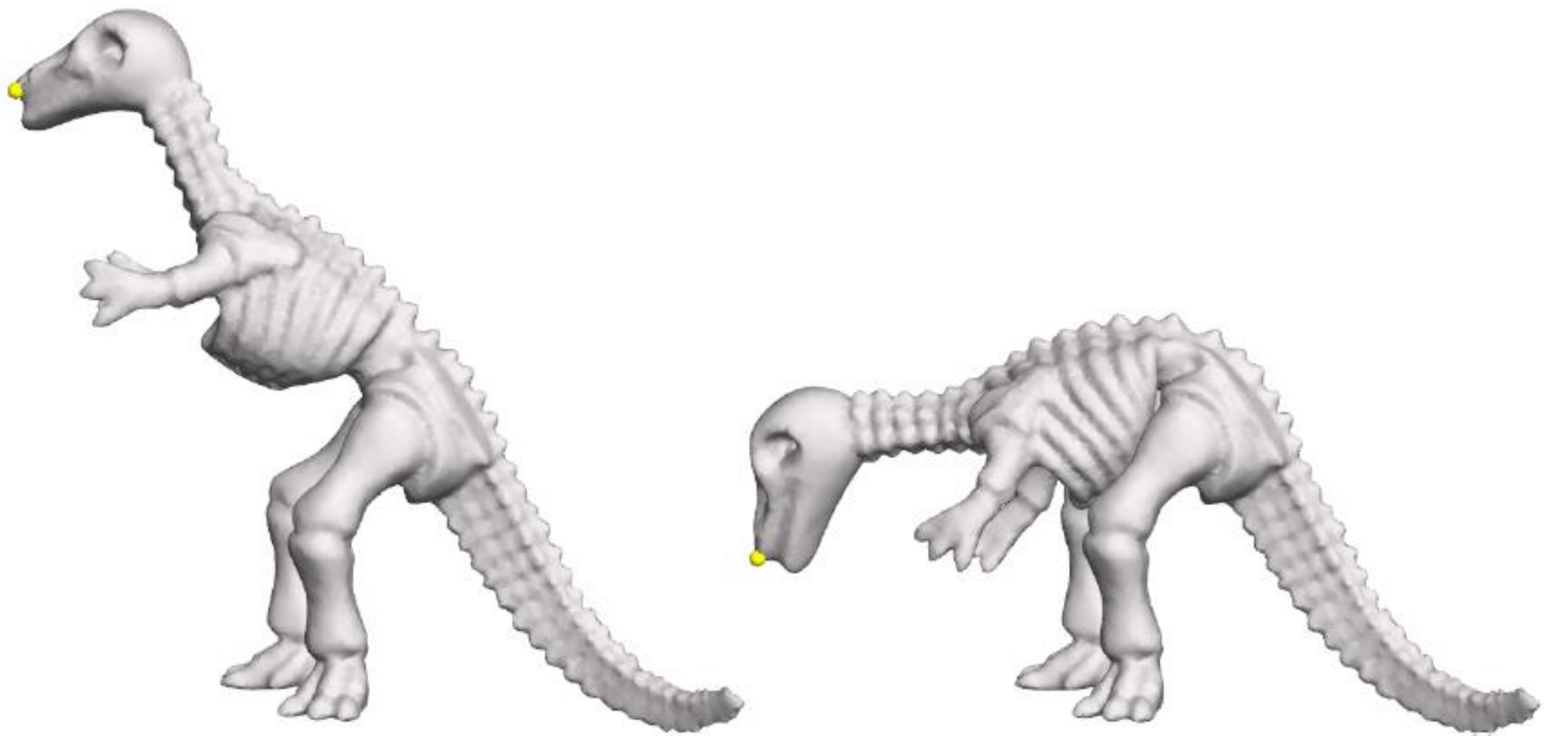
# As-Rigid-As-Possible (ARAP)

$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}\left(A_j\right) \right\|_F^2$$

amount of non-rigidity

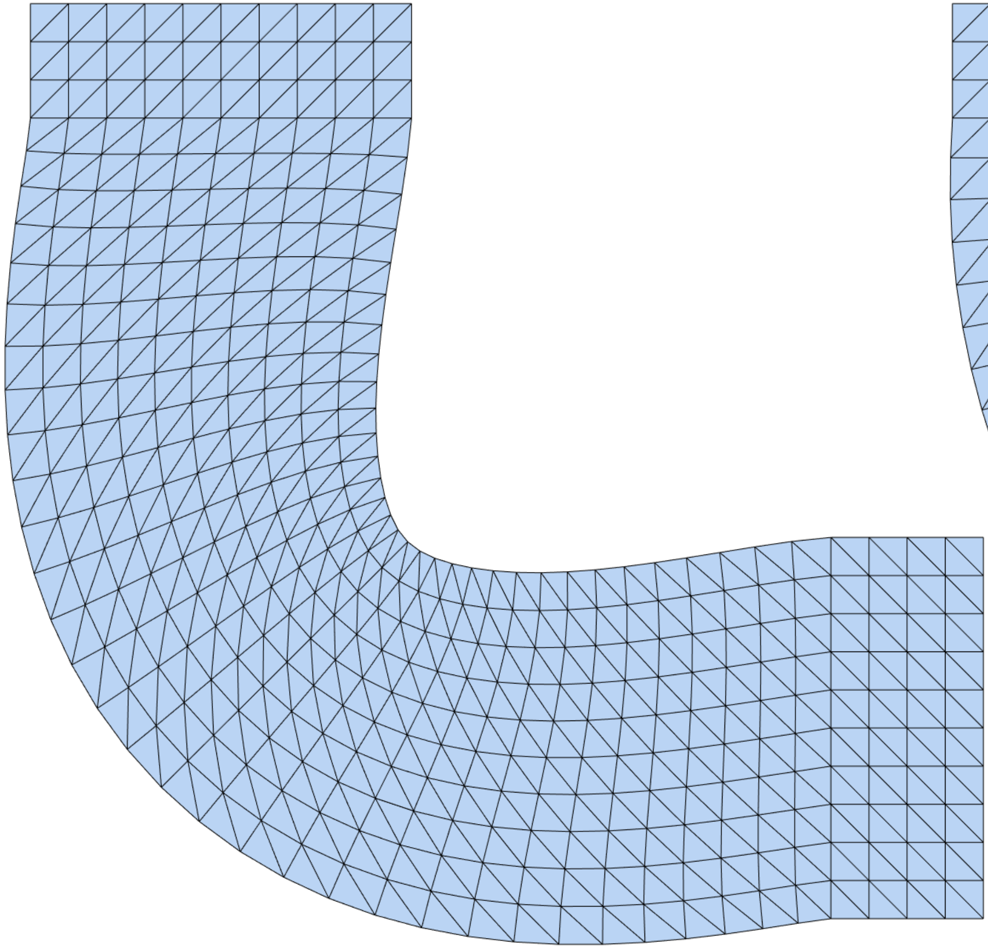


# As-Rigid-As-Possible (ARAP)

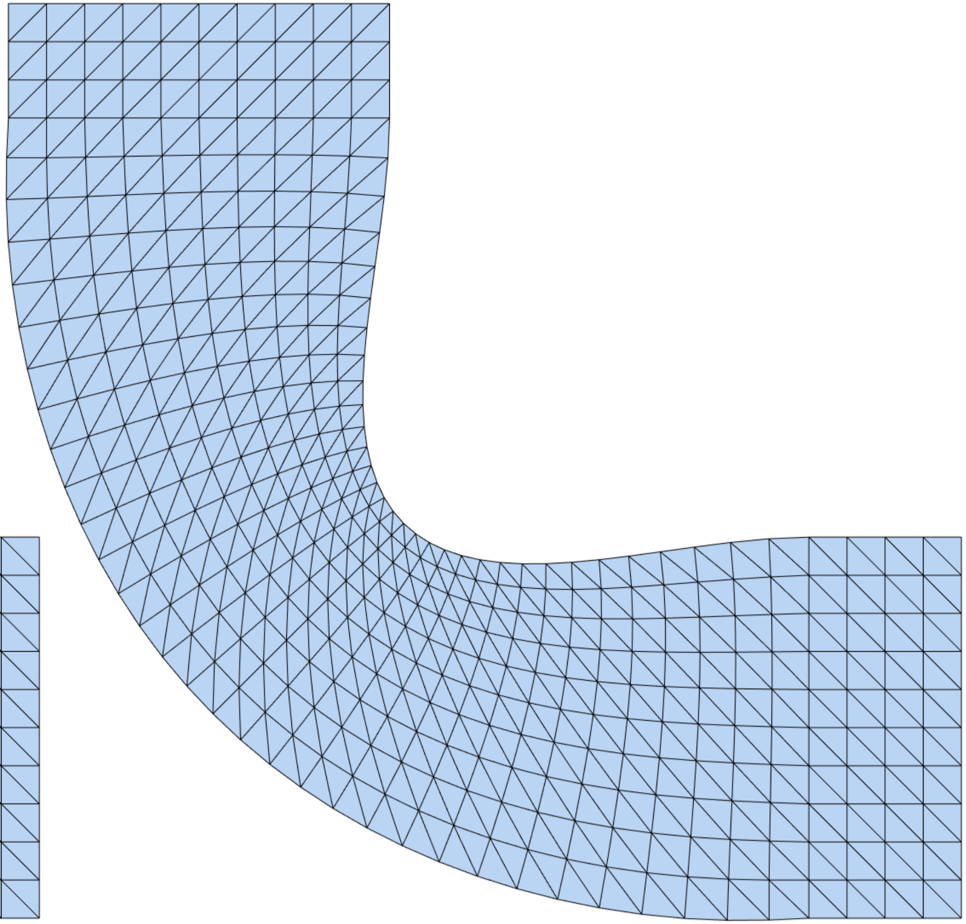


[Sorkine & Alexa 2007*; Chao et al. 2010]

# ARAP vs. LSCM

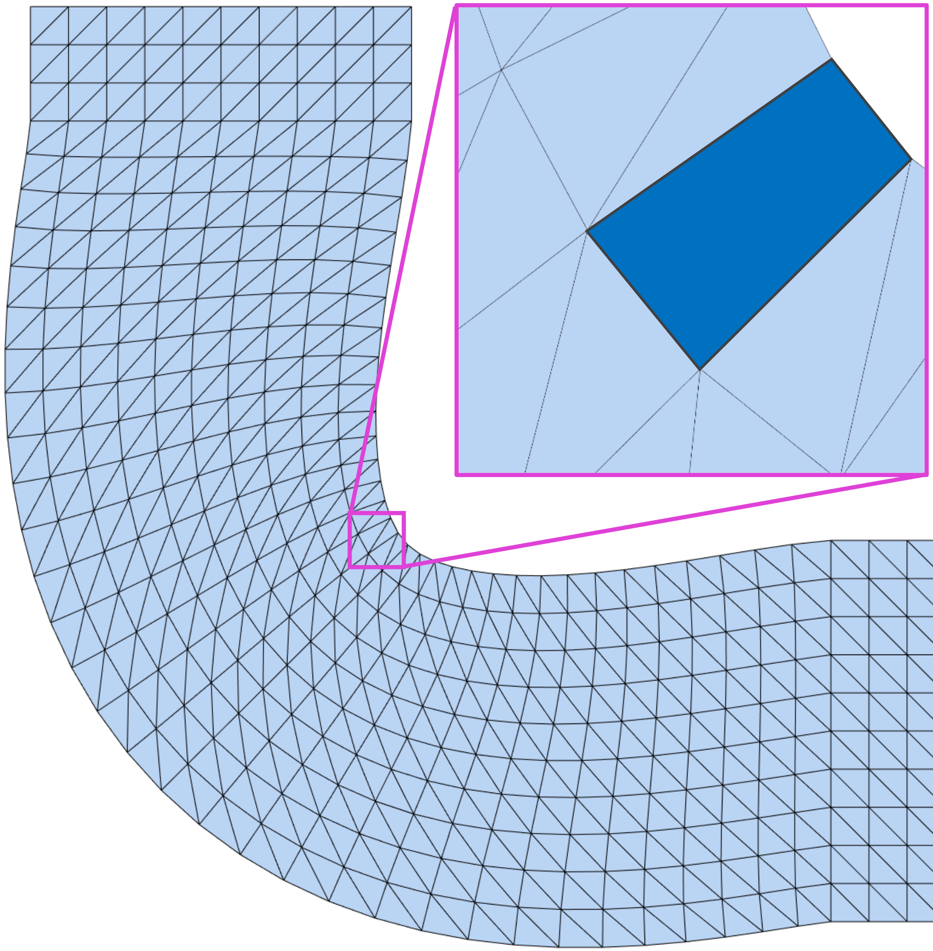


ARAP

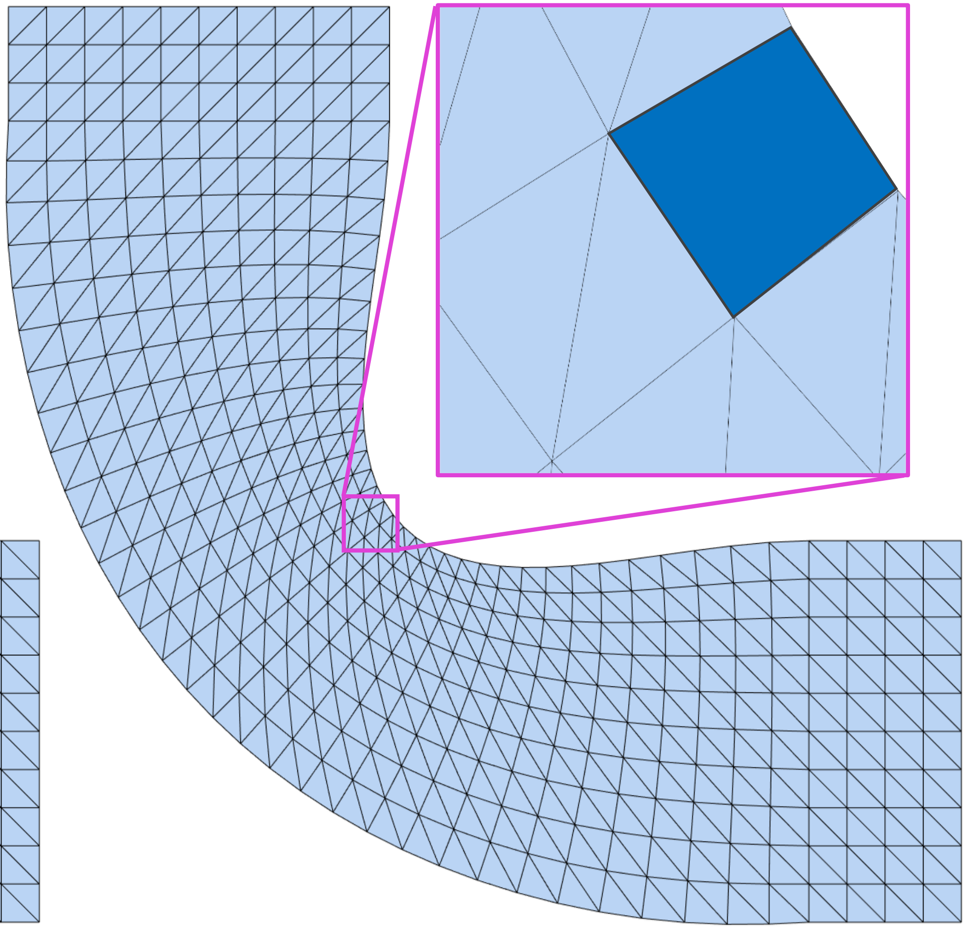


LSCM

# ARAP vs. LSCM



ARAP



LSCM

# Recap: Popular energies

**Dirichlet**

$$\|A_j\|_F^2$$

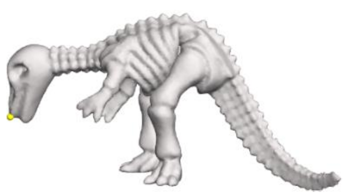
---

**LSCM**

$$\|A_j - \mathcal{S}(A_j)\|_F^2$$

---

**ARAP**



$$\|A_j - \mathcal{R}(A_j)\|_F^2$$

# Recap: Popular energies

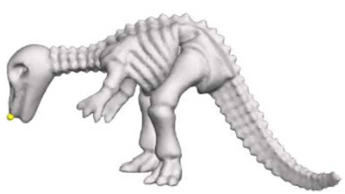
| | | | |
|---|---|---|---|
| **Dirichlet** |  | $\|A_j\|_F^2$ | **Least squares** |
| **LSCM** |  | $\|A_j - \mathcal{S}(A_j)\|_F^2$ | |
| **ARAP** |  | $\|A_j - \mathcal{R}(A_j)\|_F^2$ | |

# Closest Similarity – 2d case

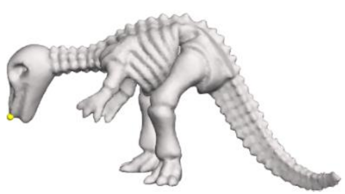
- $\mathcal{S}(A) = \overline{\sigma} U V^T$
- Takes a closed form:

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} a+d & c-b \\ b-c & a+d \end{bmatrix}}_{\text{similarity}} + \underbrace{\frac{1}{2}\begin{bmatrix} a-d & c+b \\ b+c & -a+d \end{bmatrix}}_{\text{anti-similarity}}$$

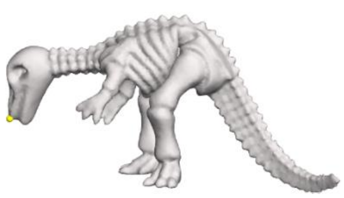
# Recap: Popular energies

| | | | |
|---|---|---|---|
| **Dirichlet** |  | $\left\Vert A_j \right\Vert_F^2$ | Least squares |
| **LSCM** |  | $\left\Vert A_j - \mathcal{S}(A_j) \right\Vert_F^2$ anti-similarity | |
| **ARAP** |  | $\left\Vert A_j - \mathcal{R}(A_j) \right\Vert_F^2$ | |

# Recap: Popular energies

| | | | |
|---|---|---|---|
| **Dirichlet** |  | $\left\|A_j\right\|_F^2$ | Least squares |
| **LSCM** |  | $\left\|A_j - \mathcal{S}(A_j)\right\|_F^2$ | **2d – Least squares** |
| **ARAP** |  | $\left\|A_j - \mathcal{R}(A_j)\right\|_F^2$ | |

# Recap: Popular energies

| | | | |
|---|---|---|---|
| **Dirichlet** |  | $\|A_j\|_F^2$ | Least squares |
| **LSCM** |  | $\|A_j - \mathcal{S}(A_j)\|_F^2$ | 2d - least squares<br>**iterative approximation** |
| **ARAP** |  | $\|A_j - \mathcal{R}(A_j)\|_F^2$ | **iterative approximation** |

# Where's the difficulty?

$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

- Not very friendly for direct minimization:

$$A - \mathcal{R}(A) = A - \boxed{UV^T}$$

via SVD of $A$

But $\mathcal{R}(A_j)$ is easy to compute…

# Alternating optimization

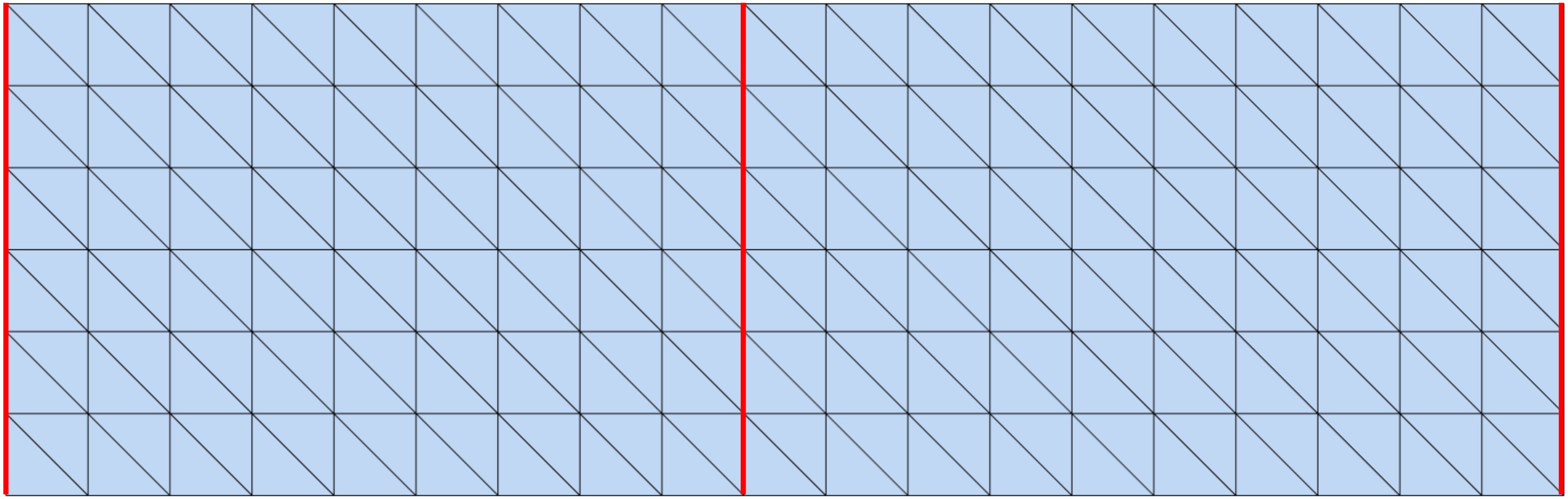
$$E_R = \sum_j w_j \left\| A_j - \mathcal{R}(A_j) \right\|_F^2$$

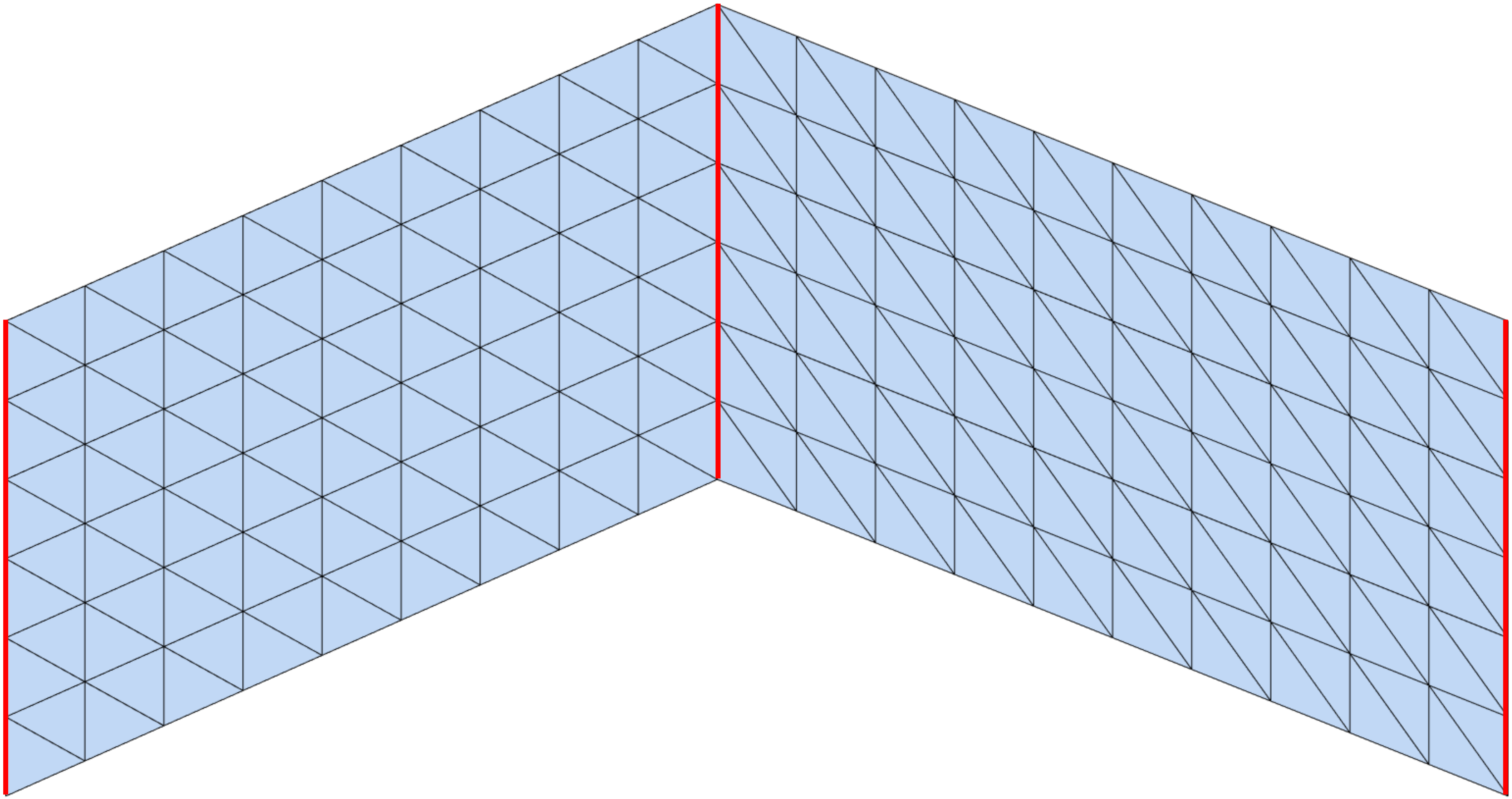
- Iteratively:
  - Compute and fix $R_j = \mathcal{R}(A_j)$ **Local** step
  - Minimize

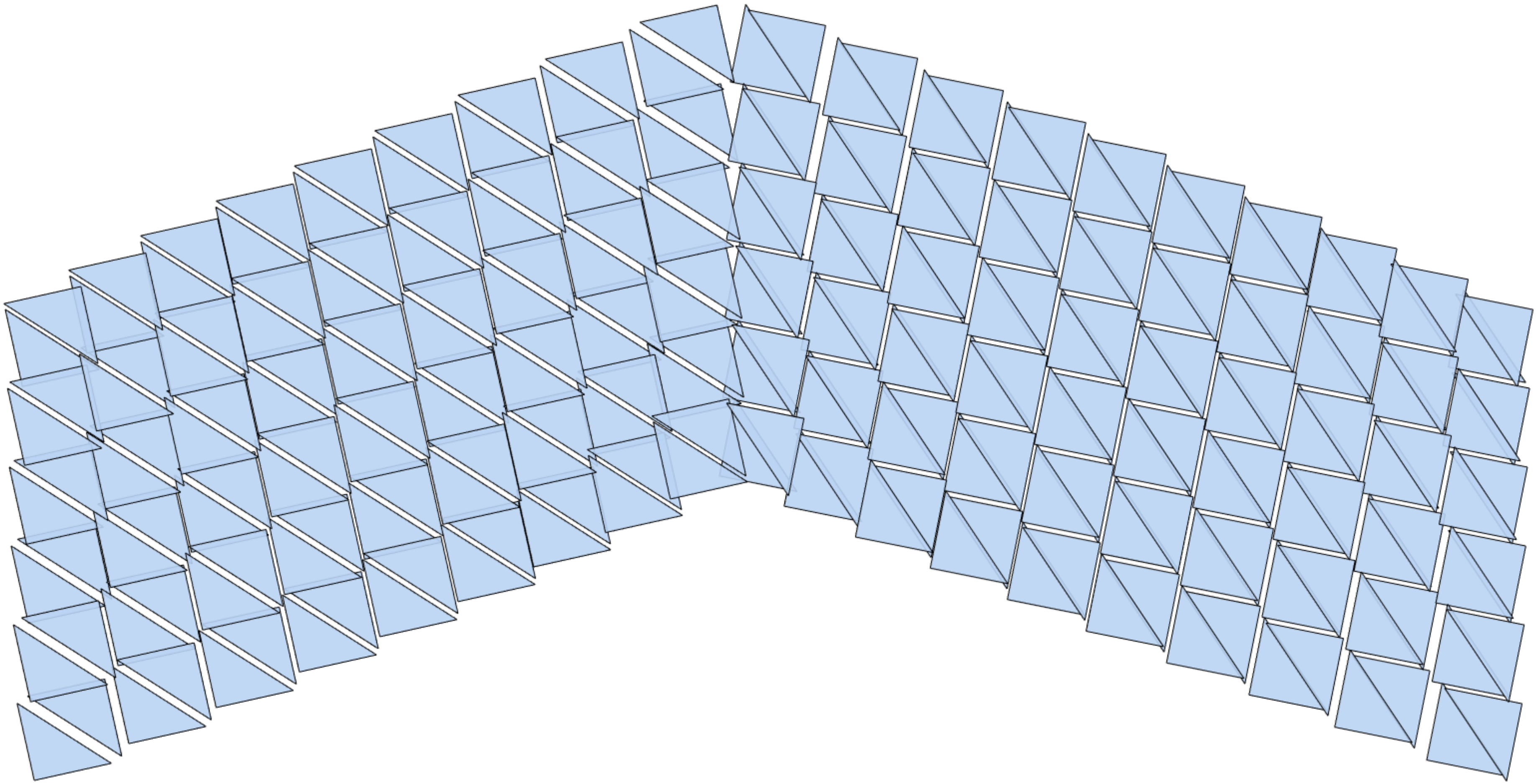
$$\sum_j w_j \left\| A_j - R_j \right\|_F^2$$

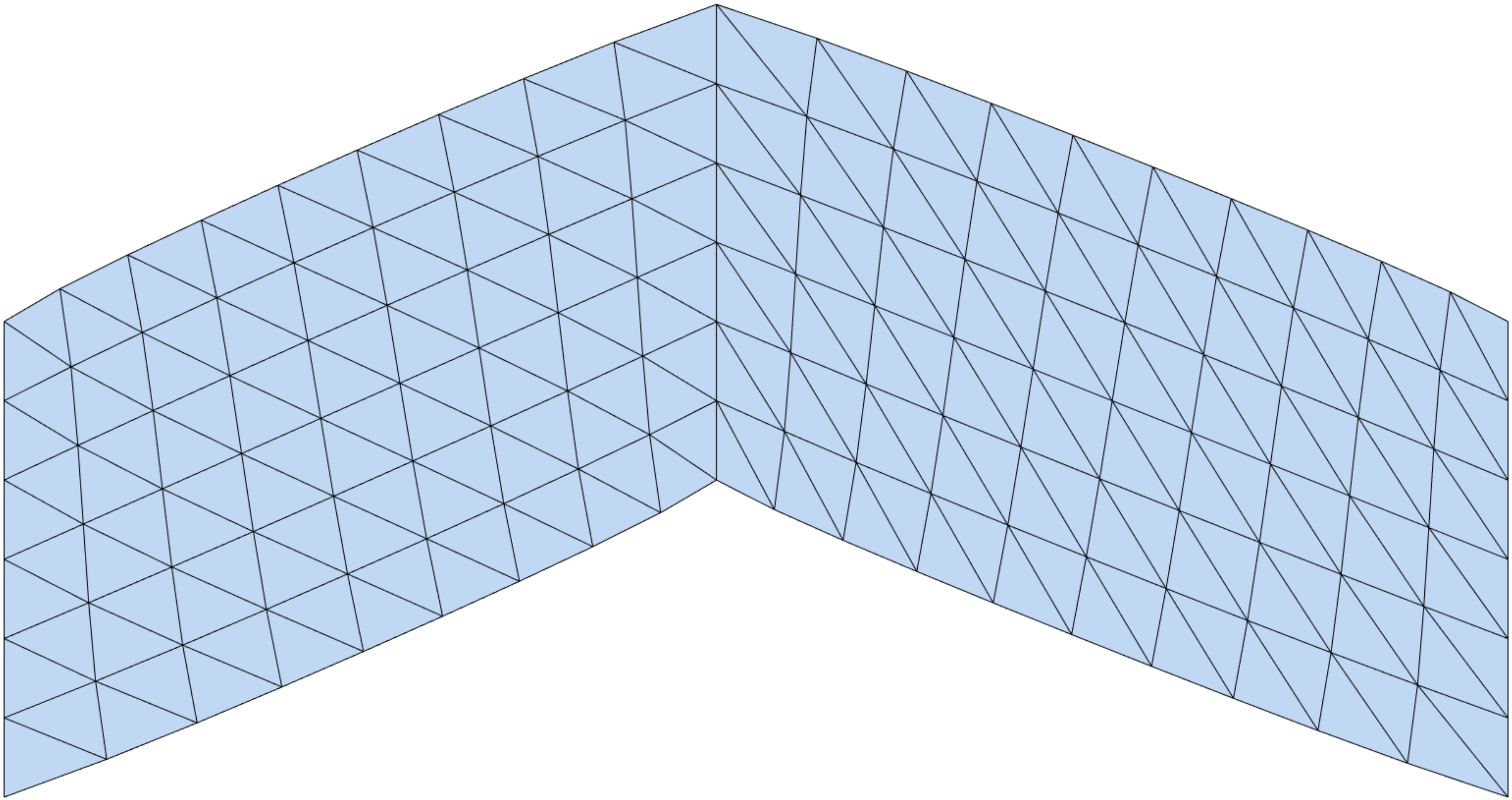
**Global** step

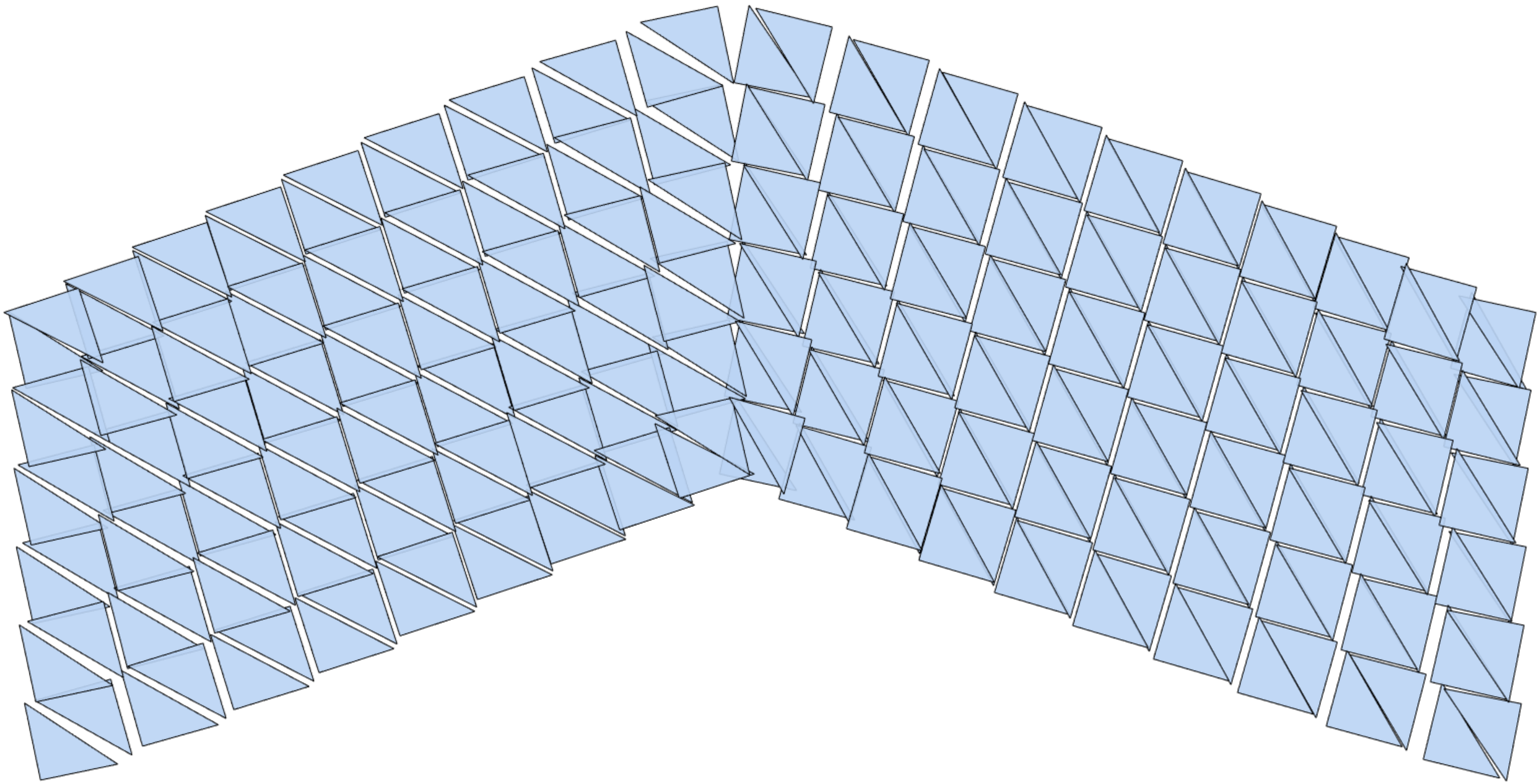
[Liu et al. 2008]

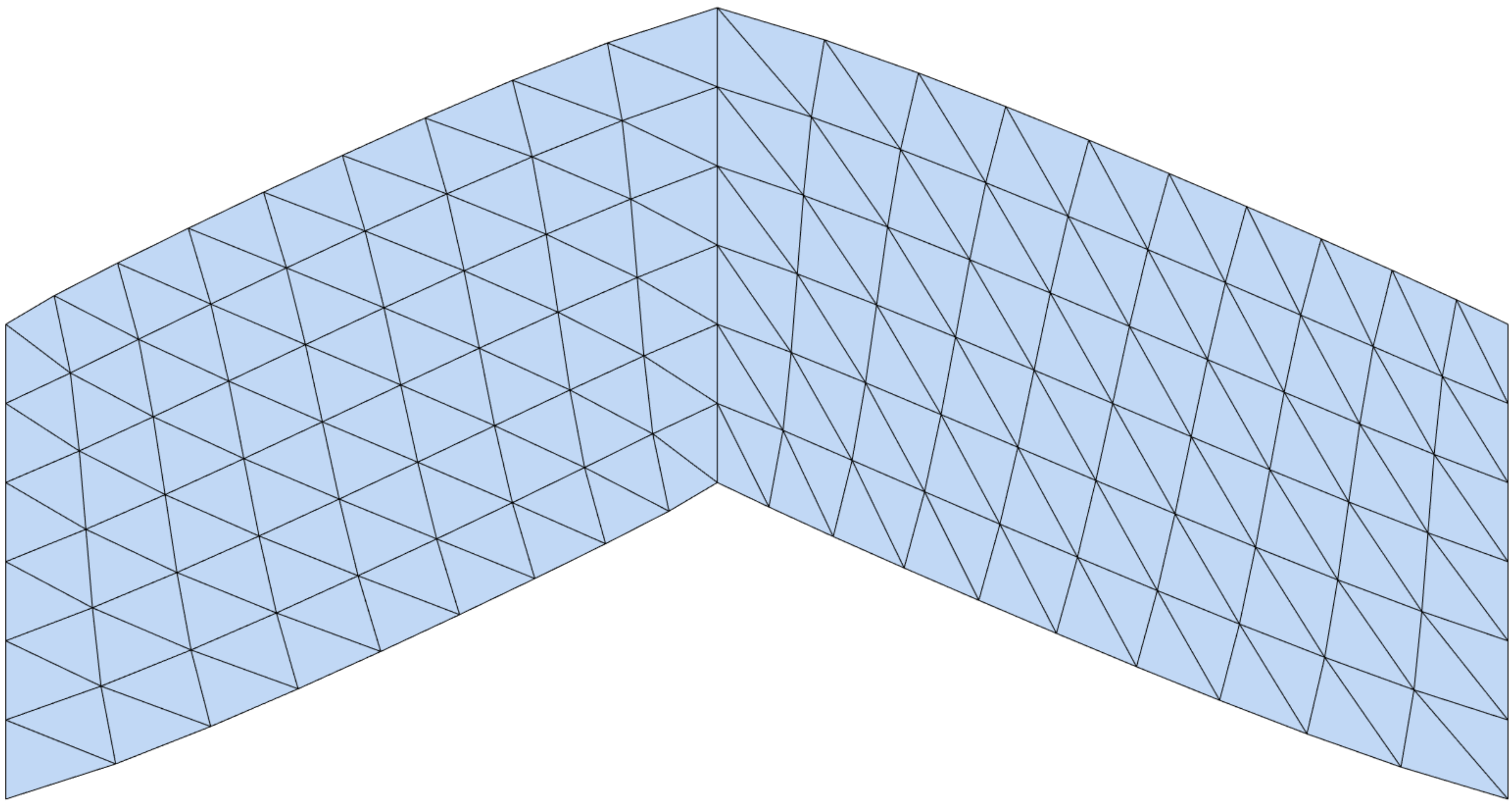


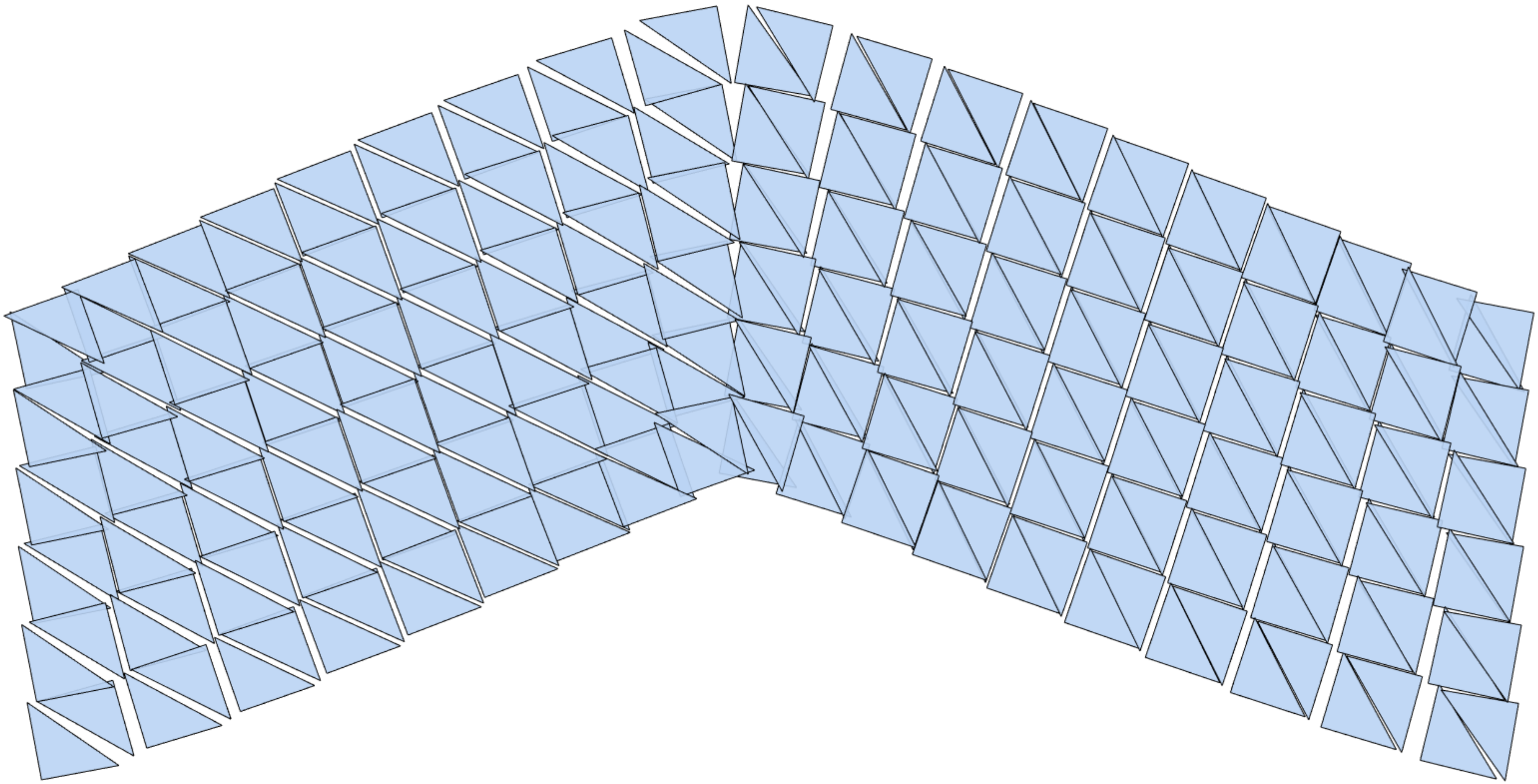


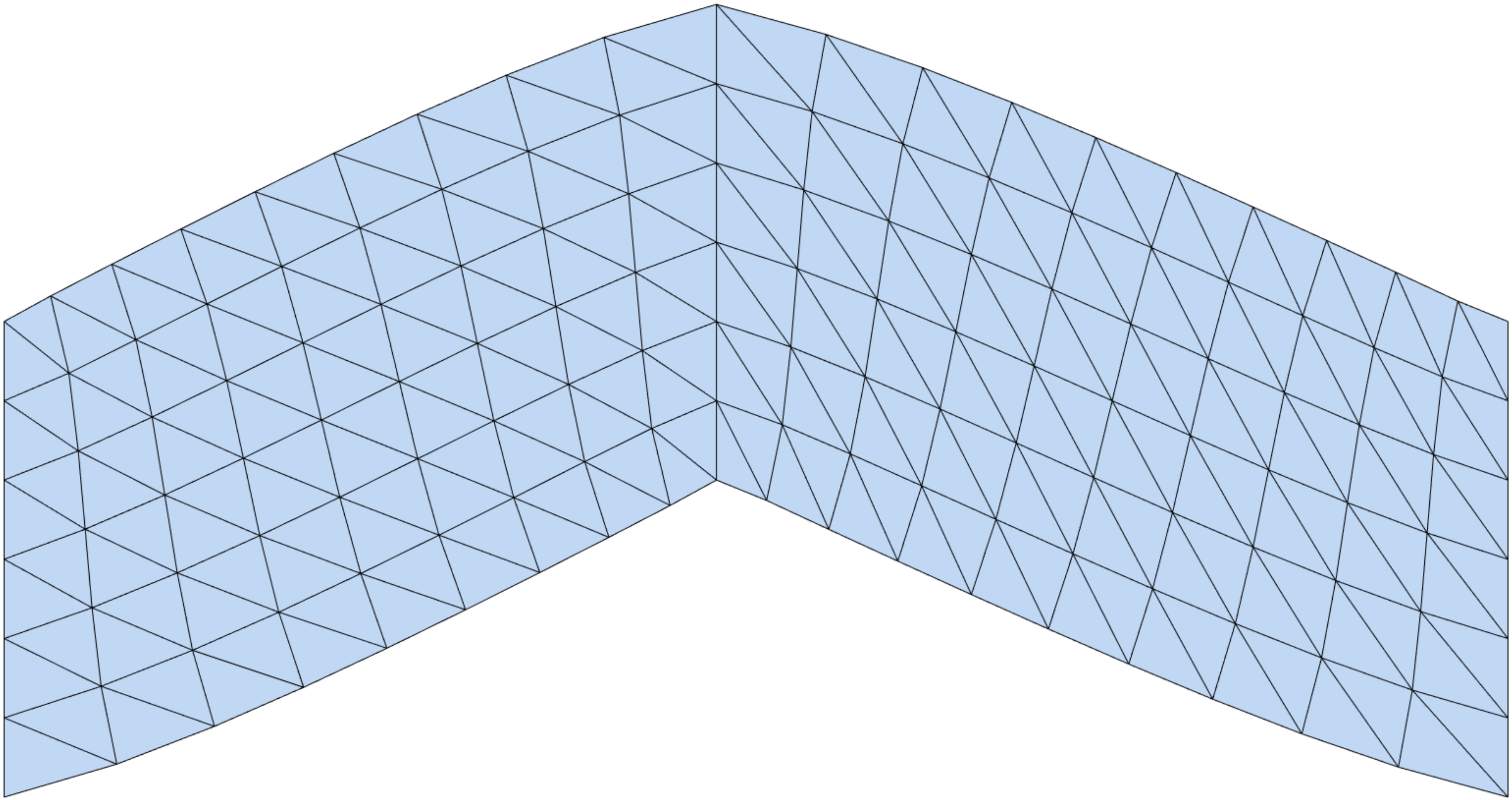


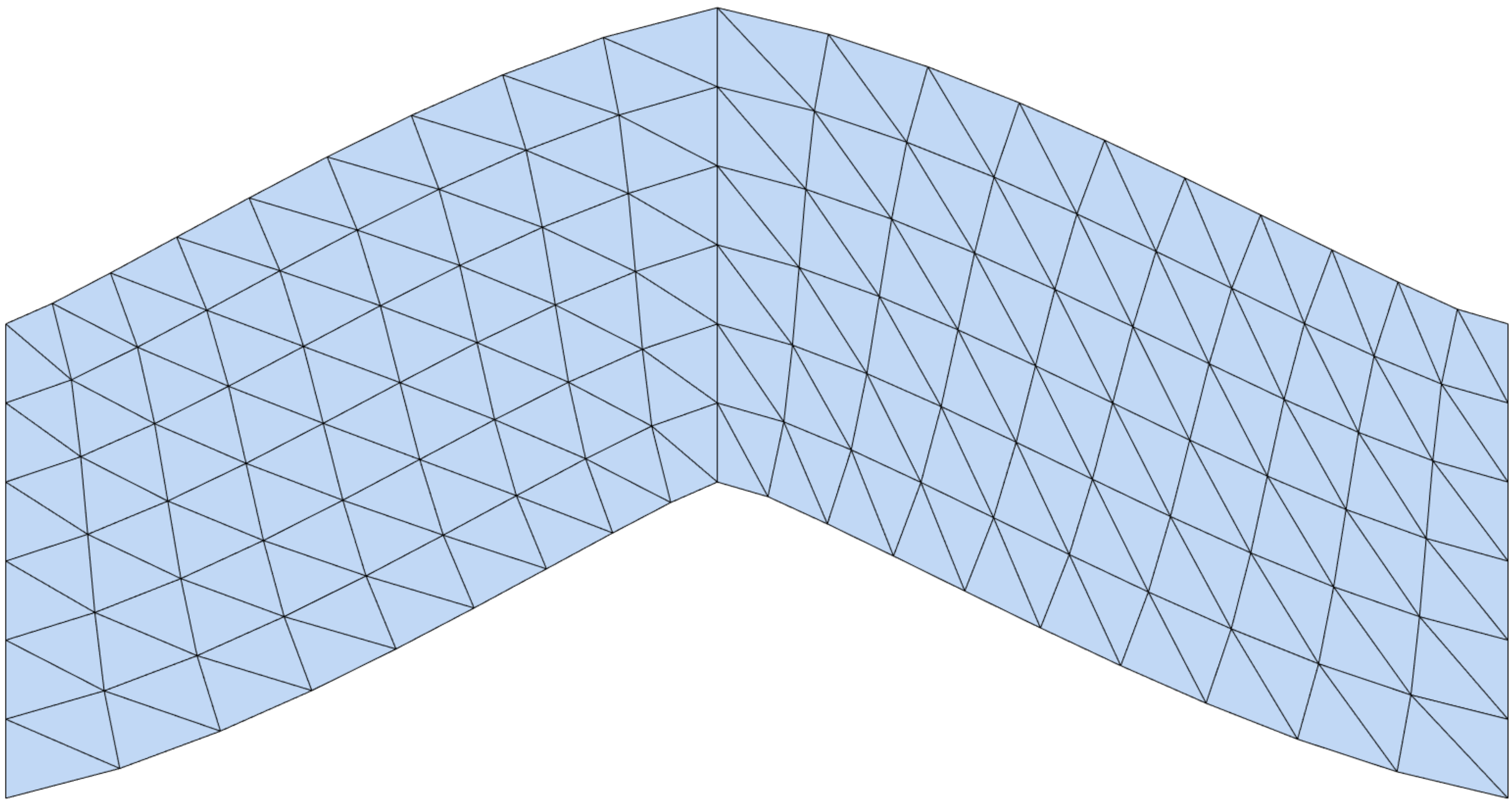


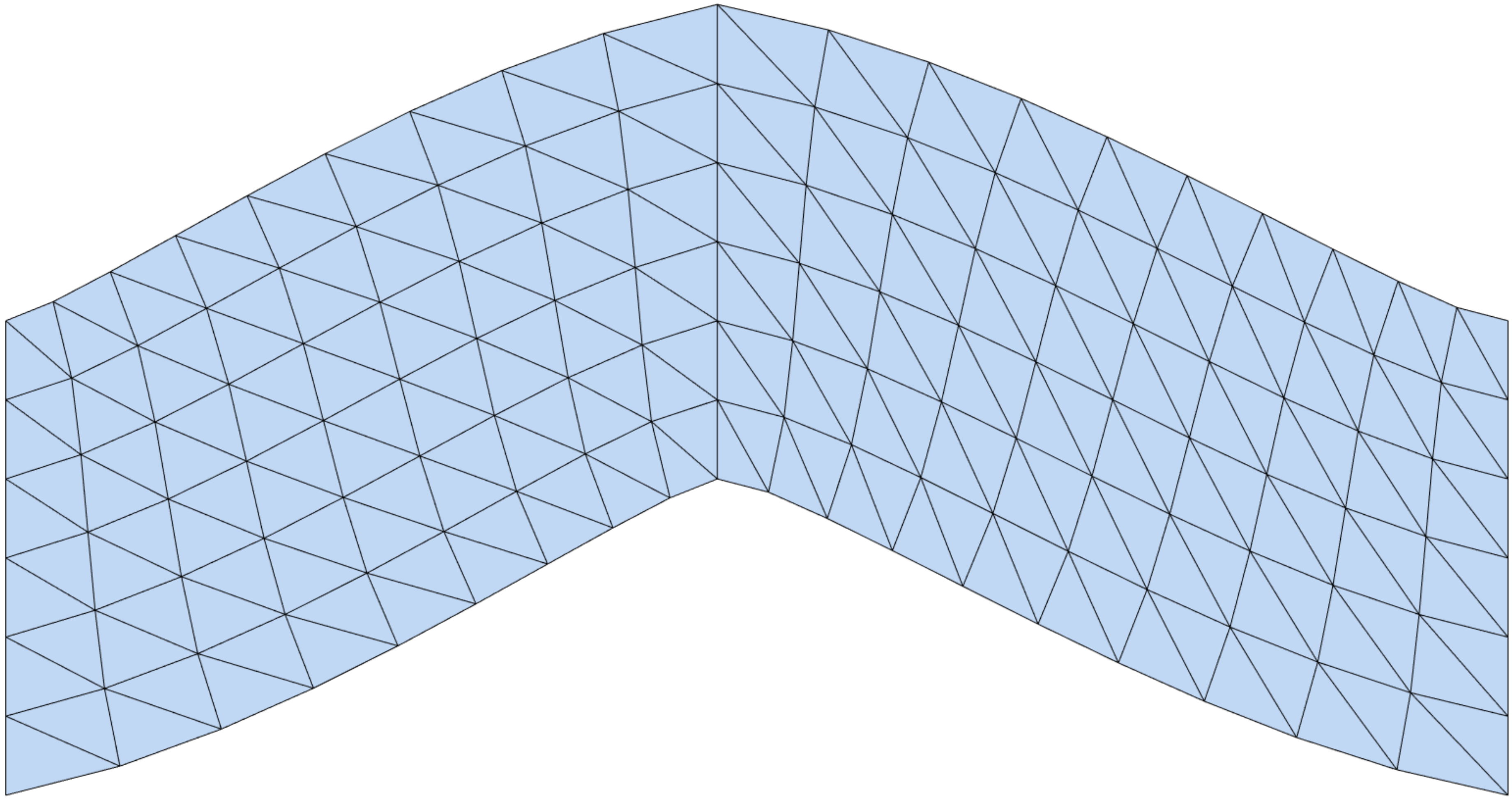


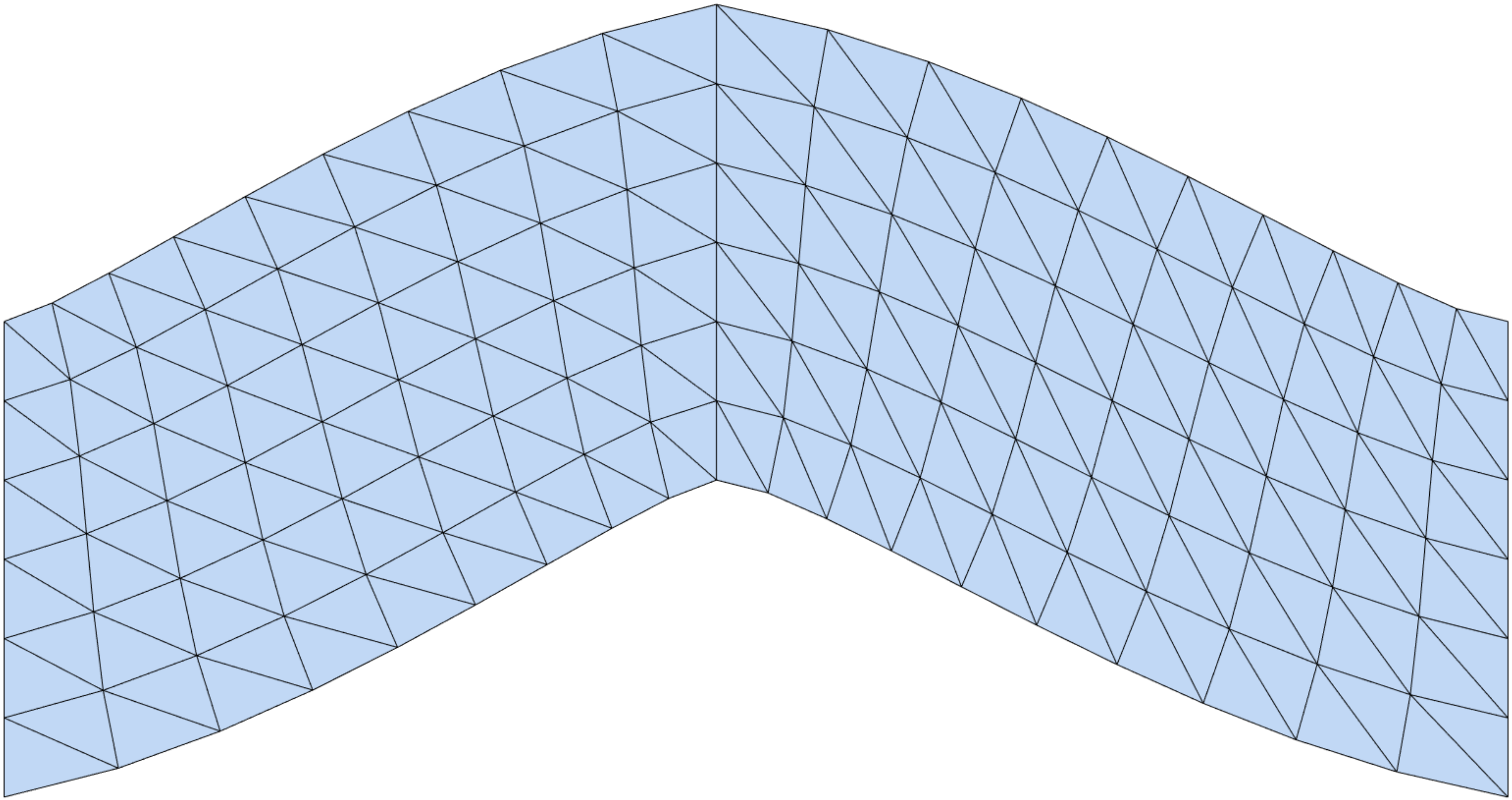


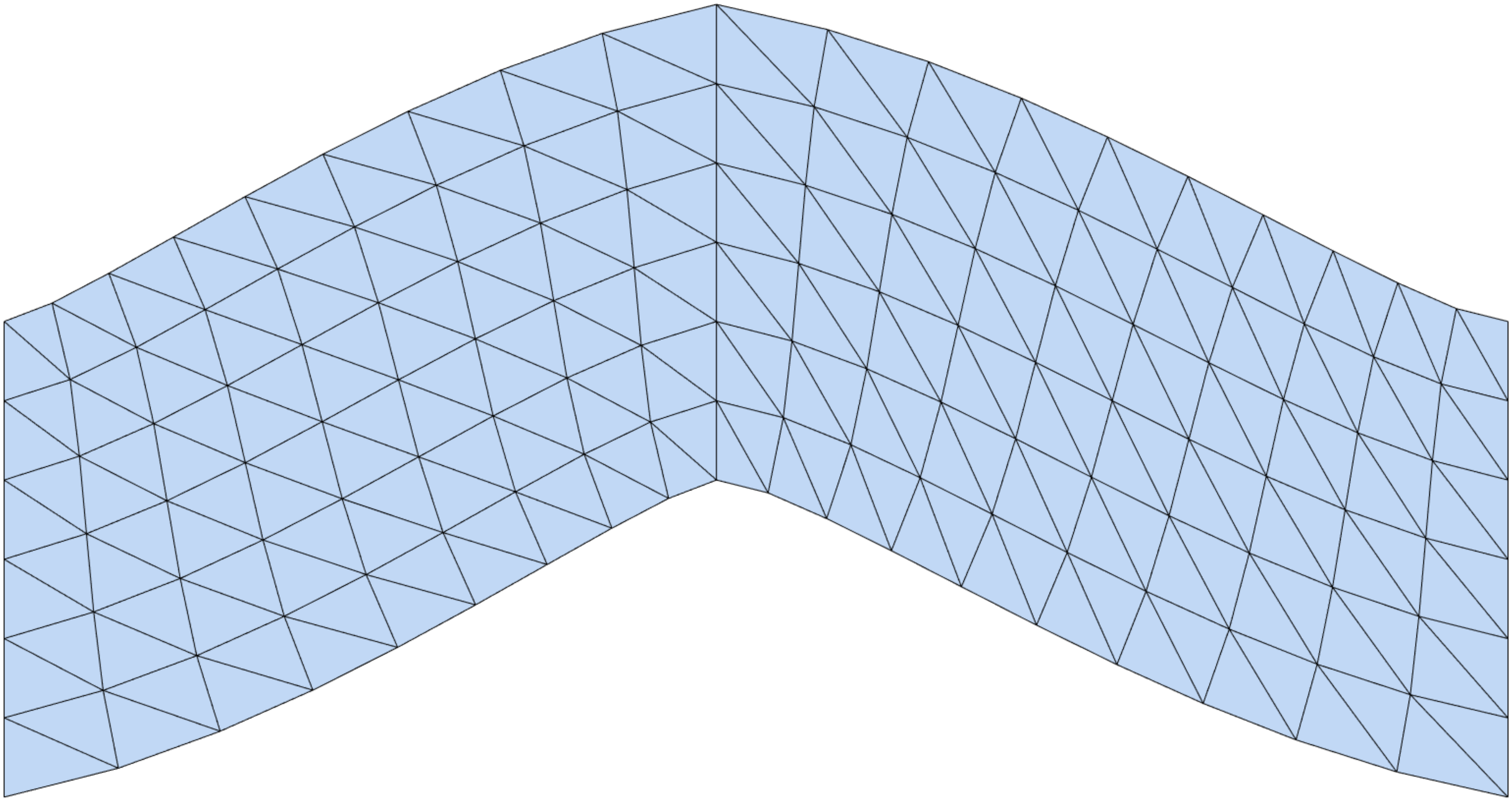






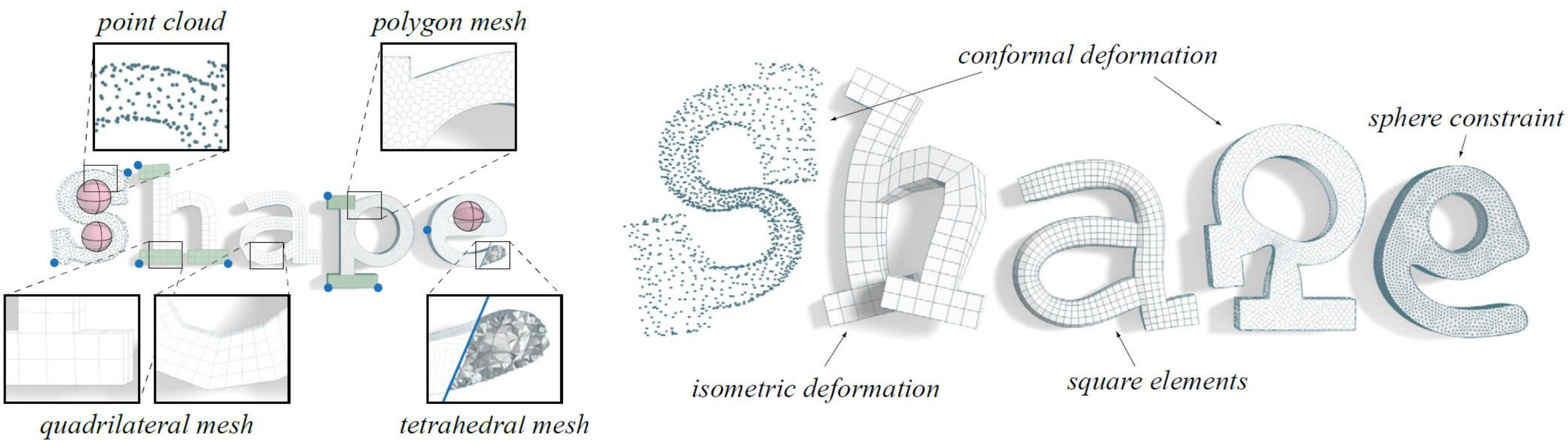






# Alternating optimization

- Very general




[Bouaziz et al. 2012]

- Related jargon:

  gradient descent, global-local,
  alternating projections, proximal algorithms

# Singular values perspective

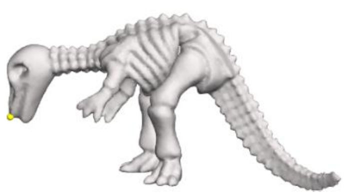
**Dirichlet**

$$\|A\|_F^2$$

**LSCM**

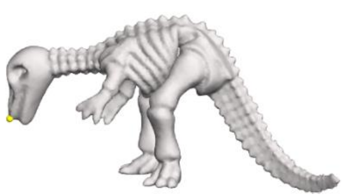
$$\|A - \mathcal{S}(A)\|_F^2$$

**ARAP**

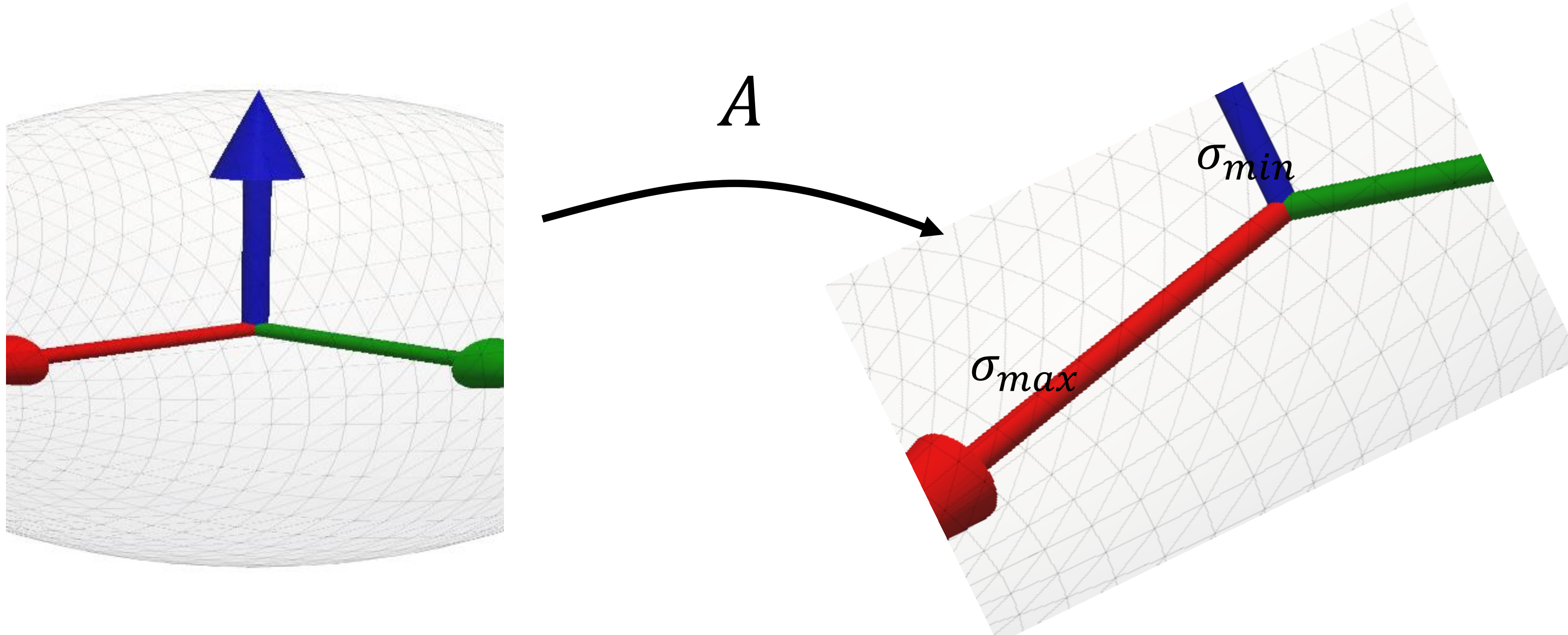


$$\|A - \mathcal{R}(A)\|_F^2$$

# Singular values perspective

| | | | |
|---|---|---|---|
| **Dirichlet** |  | $\|A\|_F^2$ | $\sum_k \sigma_k^2$ |
| **LSCM** |  | $\|A-\mathcal{S}(A)\|_F^2$ | $\sum_k (\sigma_k - \overline{\sigma})^2$ <br> mean of SVs |
| **ARAP** |  | $\|A-\mathcal{R}(A)\|_F^2$ | $\sum_k (\sigma_k - 1)^2$ |

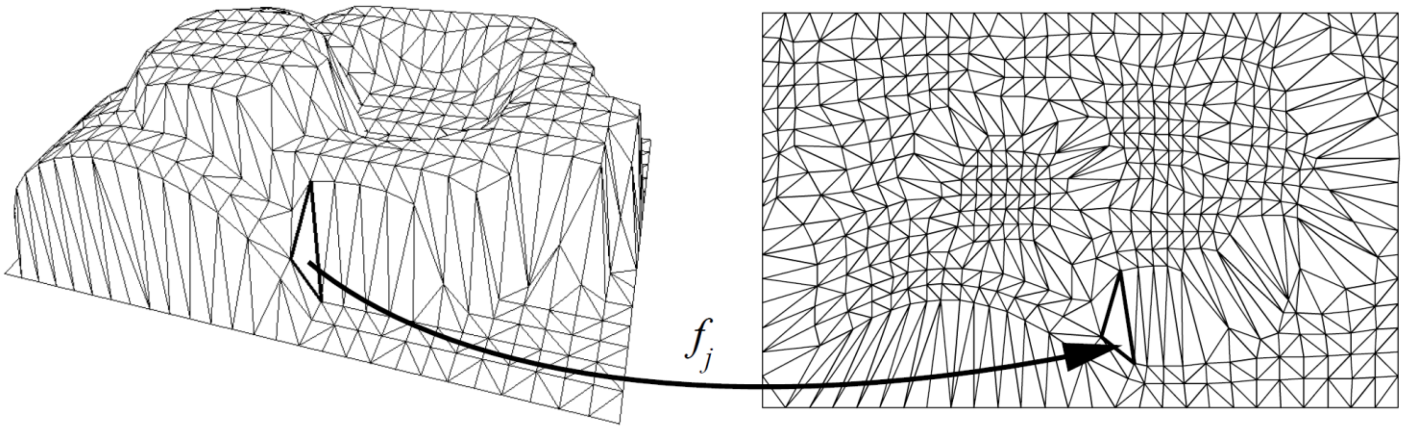
# Singular values



# Other SV energies

- Parameterization

$$\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$$

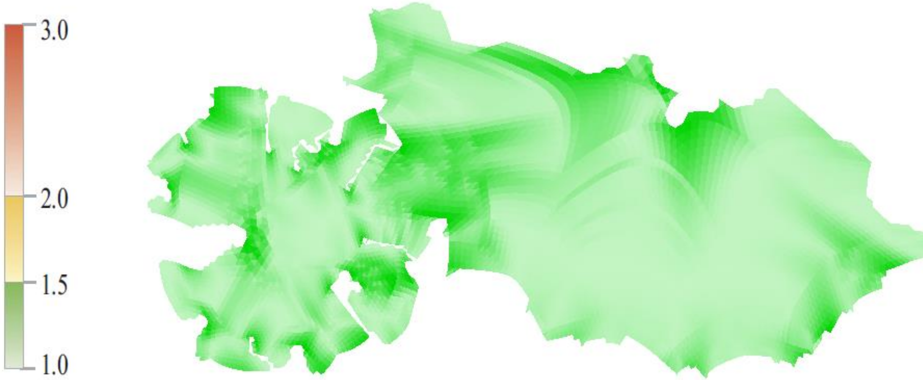


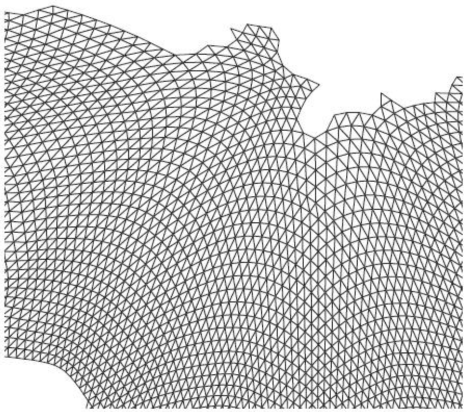

Most Isometric Parameterization
[Hormann & Greiner 2000]

# Other SV energies

- Parameterization

$$\max\left\{\sigma_1, \frac{1}{\sigma_2}\right\}$$



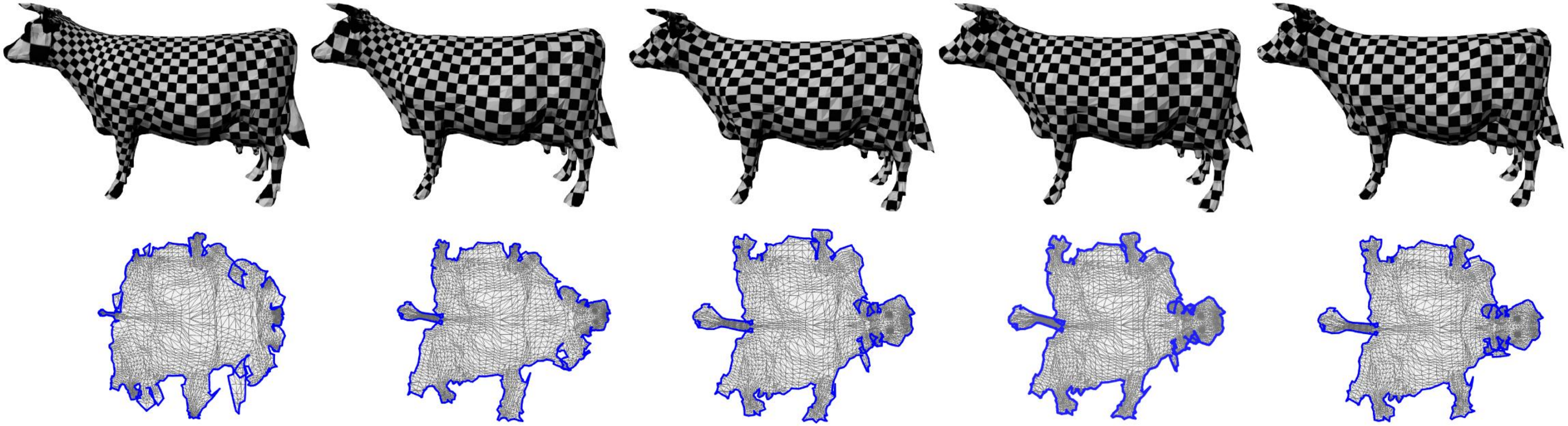



[Sorkine et al. 2000]

# Other SV energies

- Parameterization

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

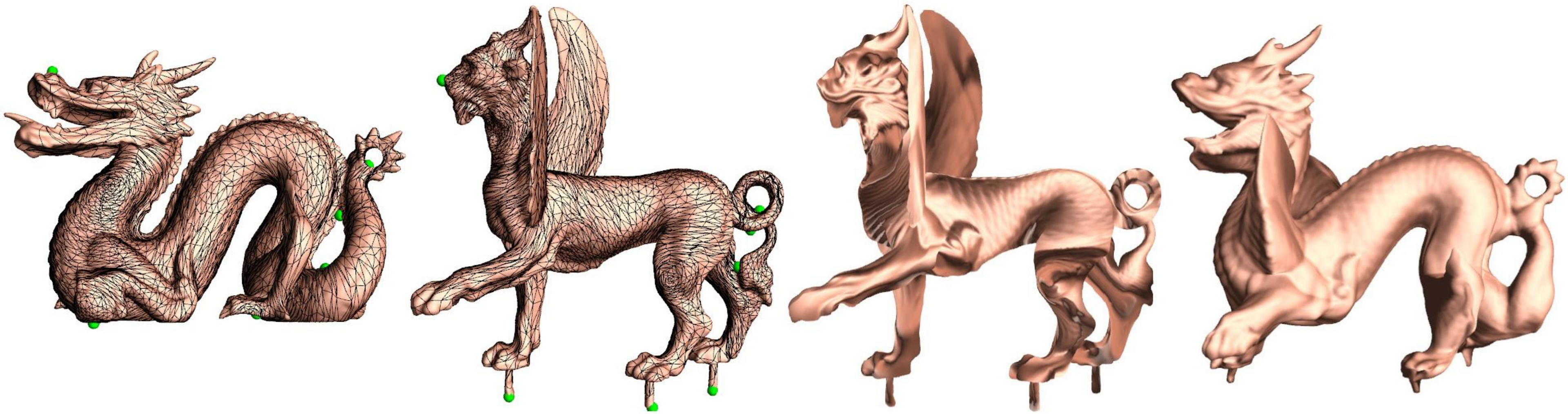


[Smith & Schaefer 2015]

# Other SV energies

- Surface mapping

$$\sigma_1^2 + \sigma_2^2 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

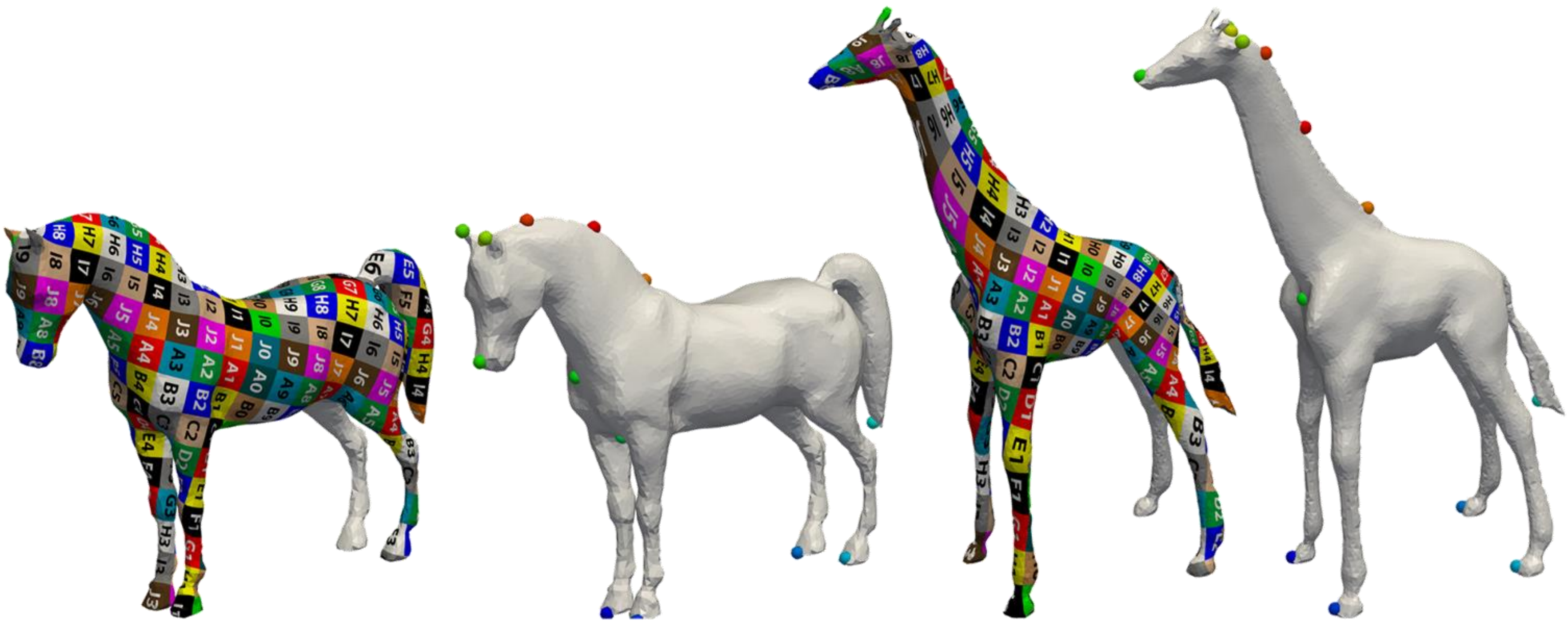


[Schreiner et al. 2014]

# Other SV energies

- Surface mapping

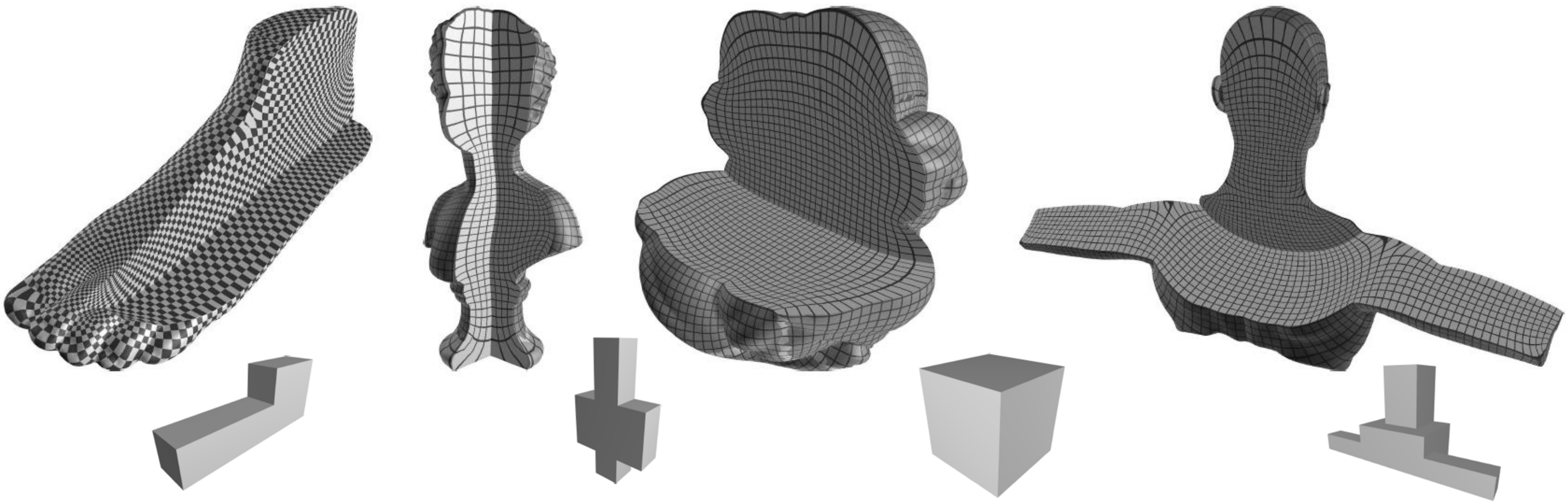
$$\sqrt{\sigma_1^2 + \frac{1}{\sigma_2^2}}$$



[Aigerman et al. 2014]

# Other SV energies

- Volume mapping $(\sigma_1 - \overline{\sigma})^2 + (\sigma_2 - \overline{\sigma})^2 + (\sigma_3 - \overline{\sigma})^2$

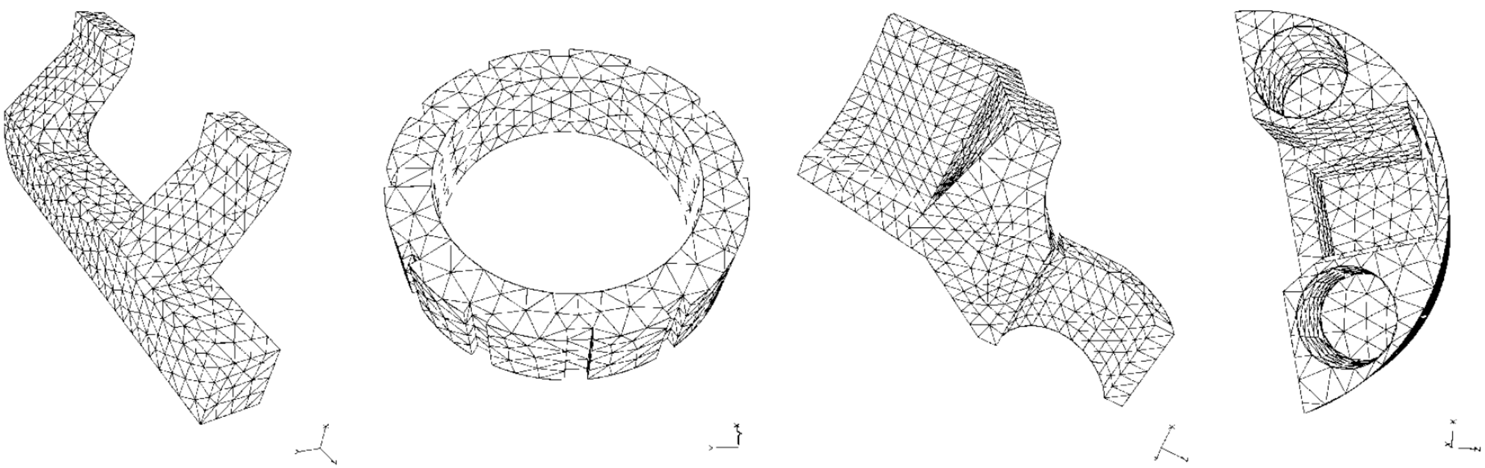


[Paillé & Poulin 2012]

# Other SV energies

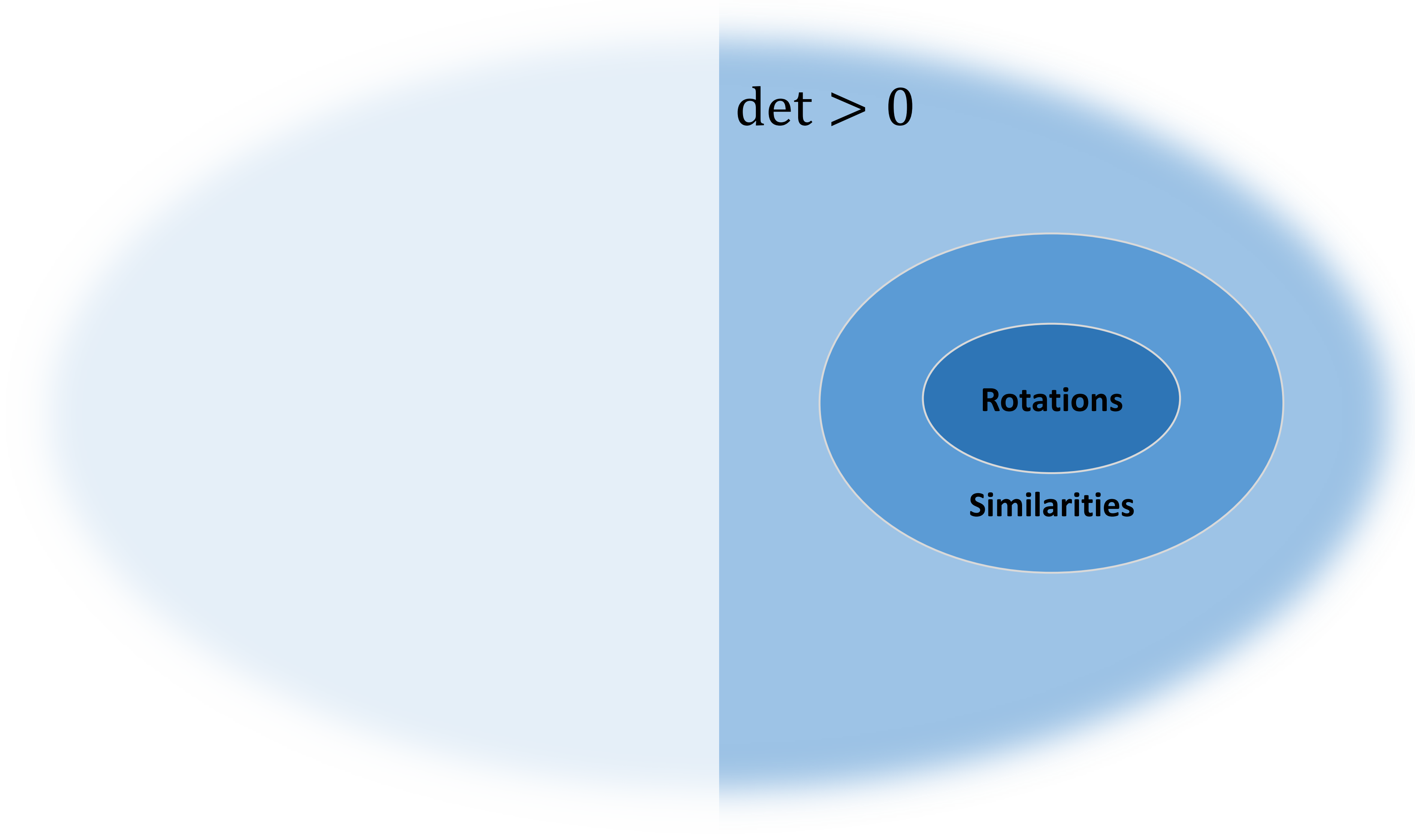
- Volumetric mesh improvement

$$\frac{\sigma_1}{\sigma_3}$$

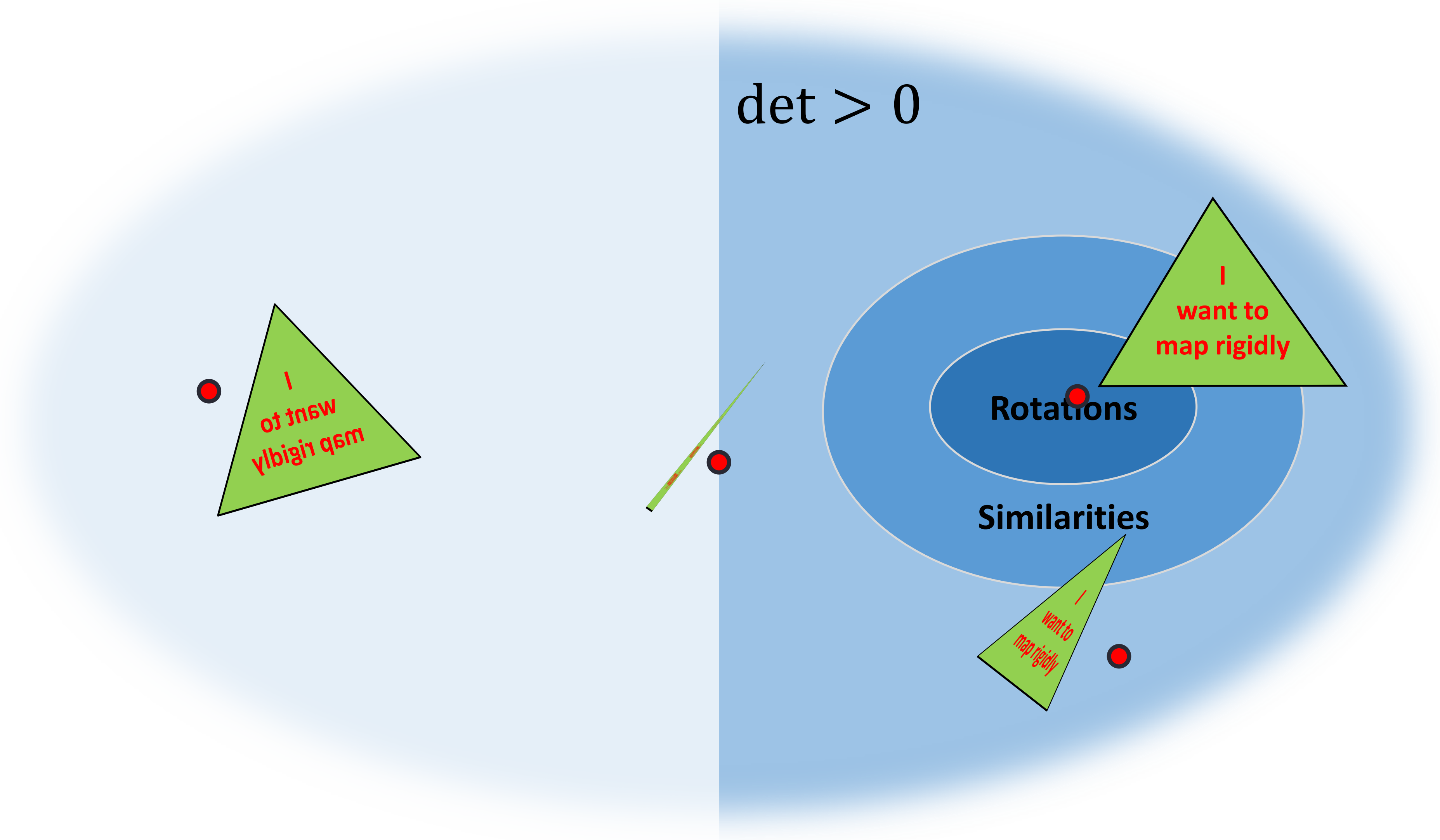


[Freitag & Knupp 2002]

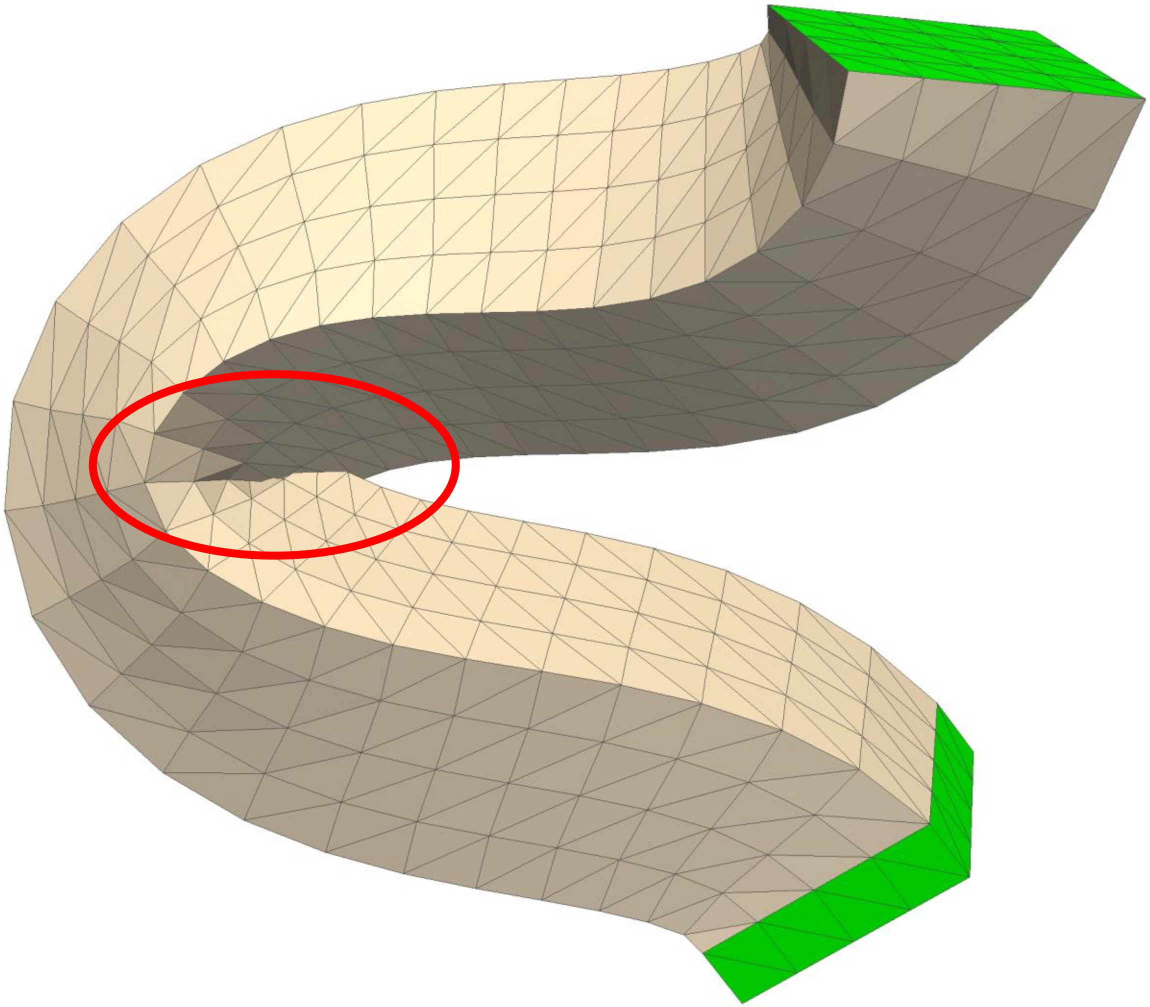
# Spaces of Mappings



# Spaces of Mappings

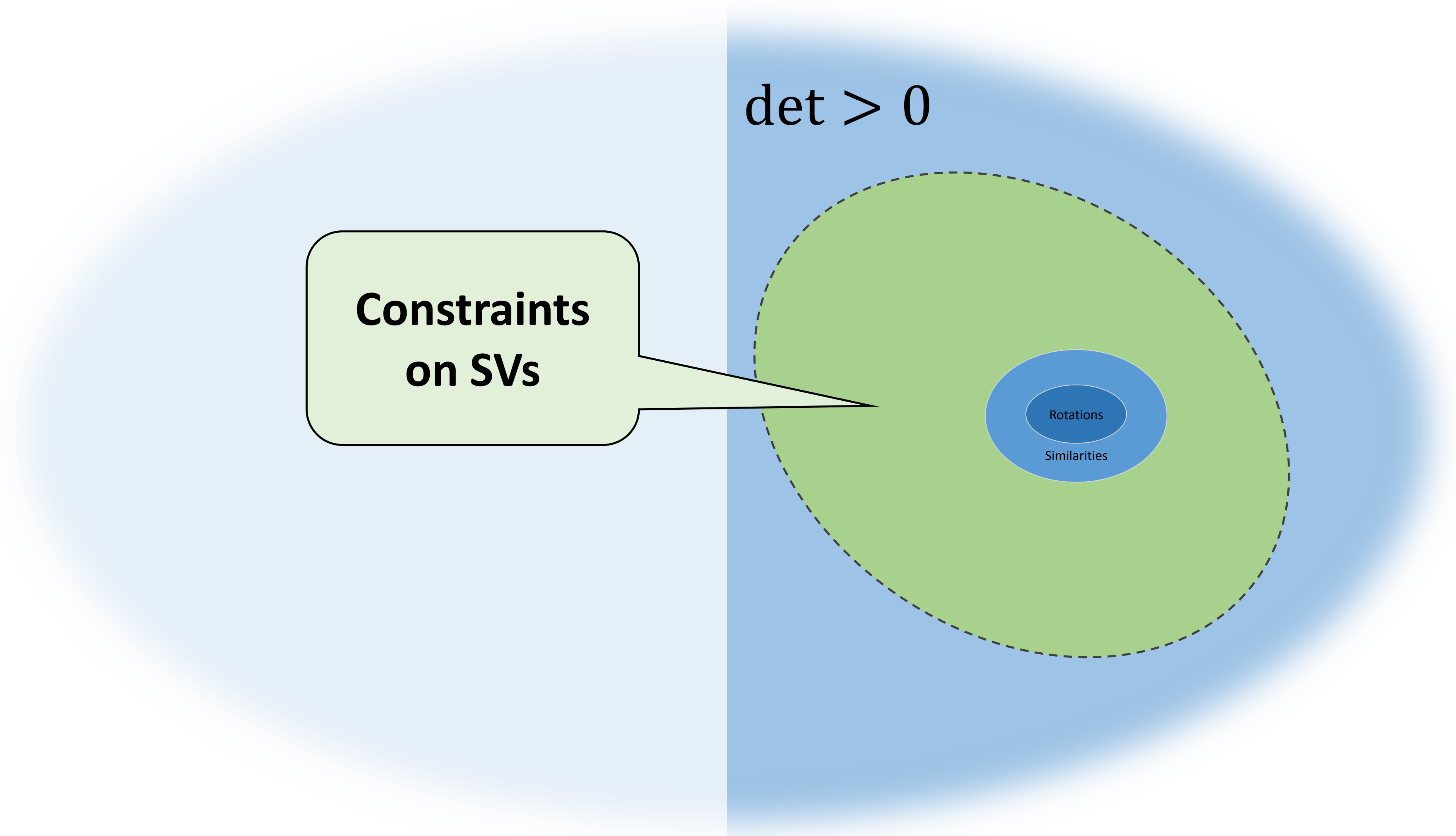


# Example



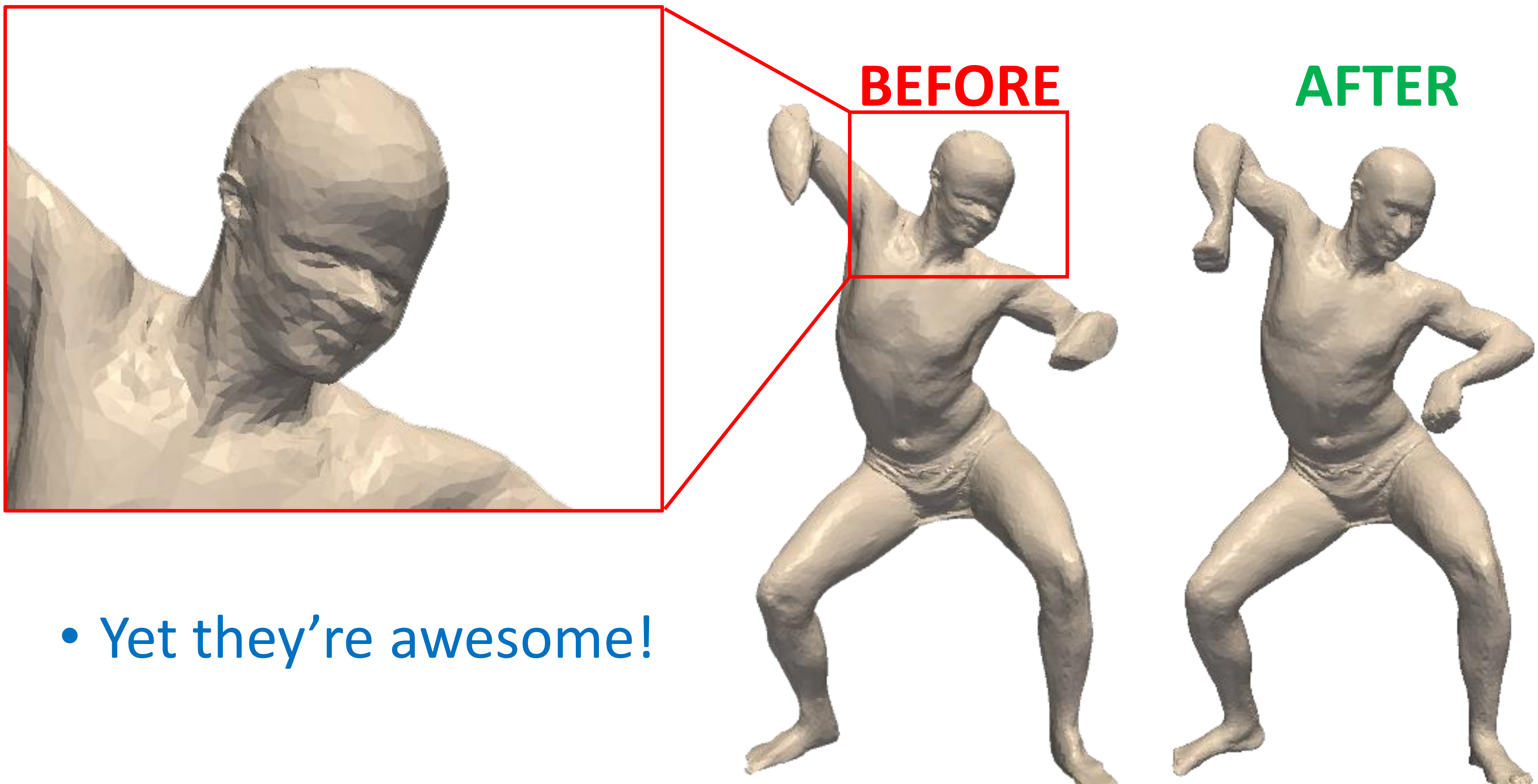
As-Rigid-As-Possible

# Spaces of Mappings



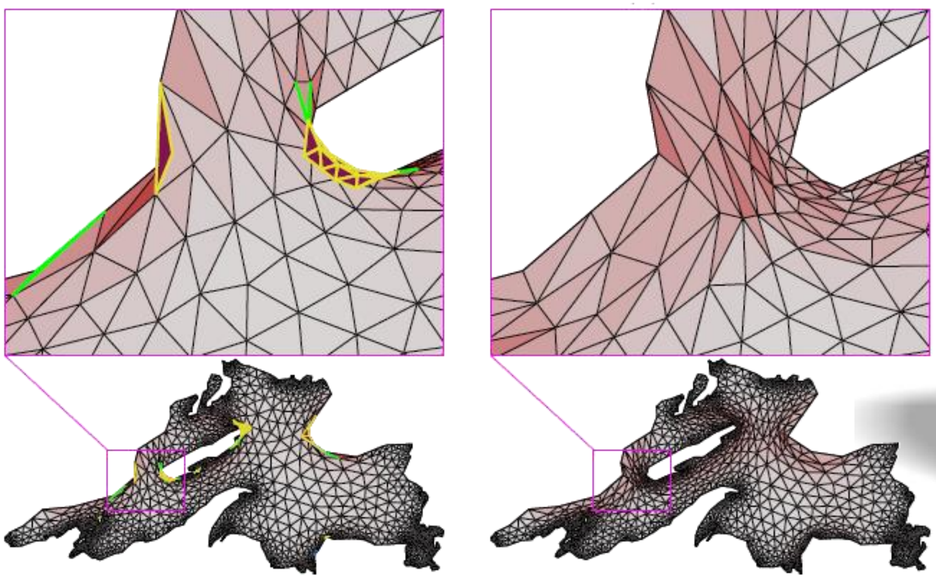
# Where's the challenge?

- Singular values = roots of polynomials

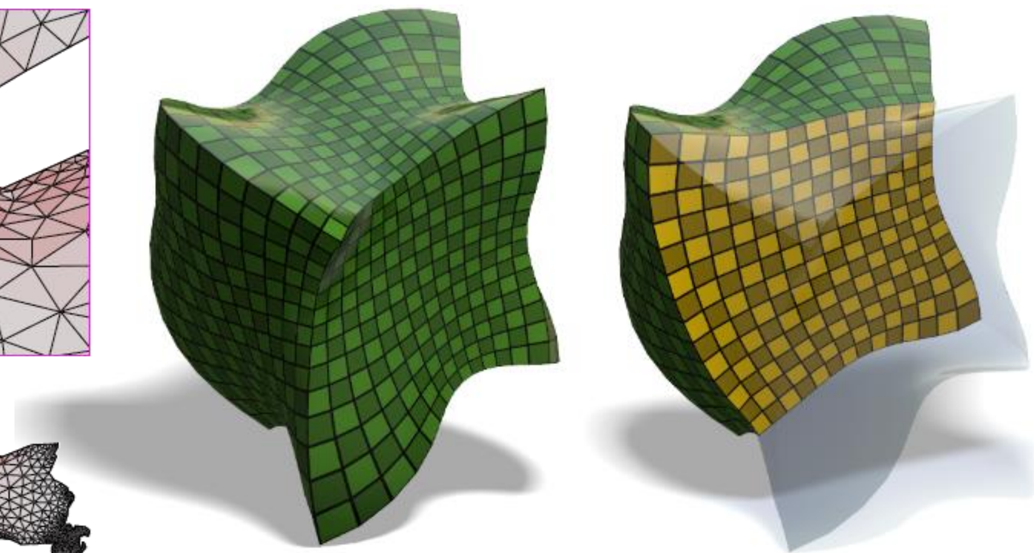



- Yet they're awesome!

# SV constraints (+energy)



[Lipman 2012]



[Kovalsky et al. 2014]

Approximate via a sequence of convex programs

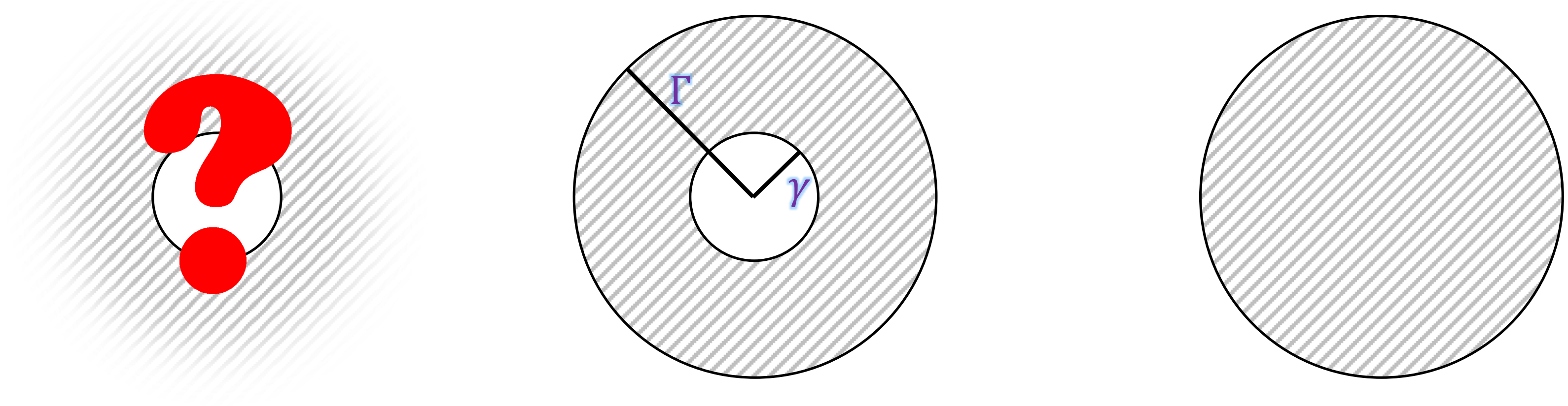
# Bounding SVs

- Simplest constraint

$$\boldsymbol{\gamma} \leq \sigma_i \leq \boldsymbol{\Gamma}$$

Non-convex

Convex



# Convex = Simple?

- Cone of positive semidefinite matrices

$$\{A : x^T A x \geq 0 \quad \text{for all} \quad x\}$$

**“easy”**

# Convex = Simple?

- Cone of copositive matrices

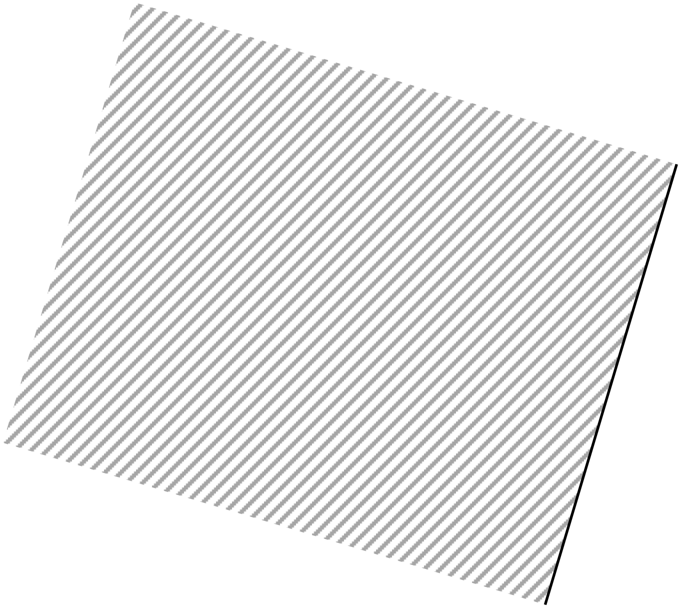
$$\{A : x^T A x \geq 0 \quad \text{for all} \quad x \geq 0\}$$

**"very difficult"**

# Standard convex conic programs

- Linear inequalities

$$c^T \mathrm{x} \leq d$$

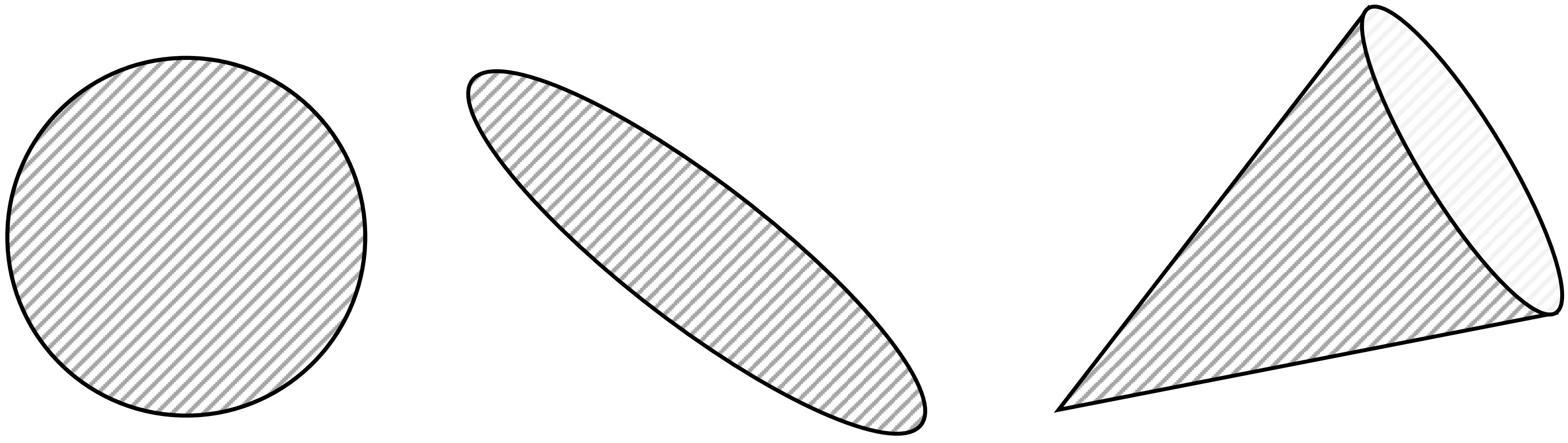


$\Rightarrow$ linear programming (LP)

# Standard convex conic programs

- Second order (ice cream) cones

$$\|x\|_2 \leq t$$



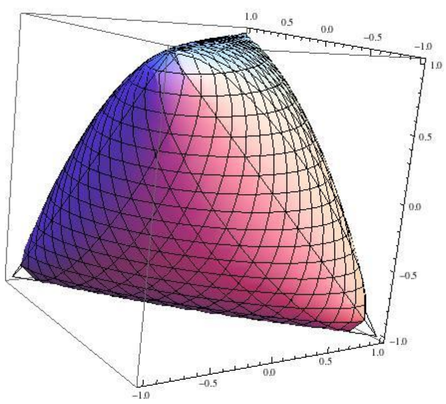
$\Rightarrow$ second order cone programming (SOCP)

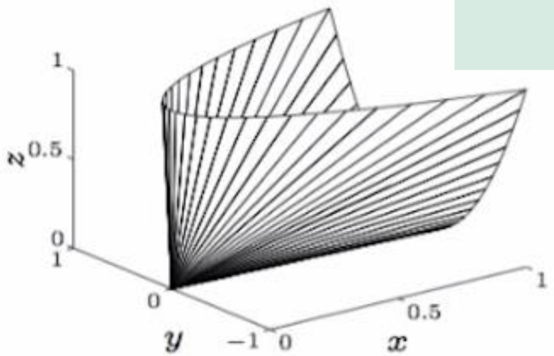
# Standard convex conic programs

- Linear matrix inequalities (LMIs)

$X \succcurlyeq 0$

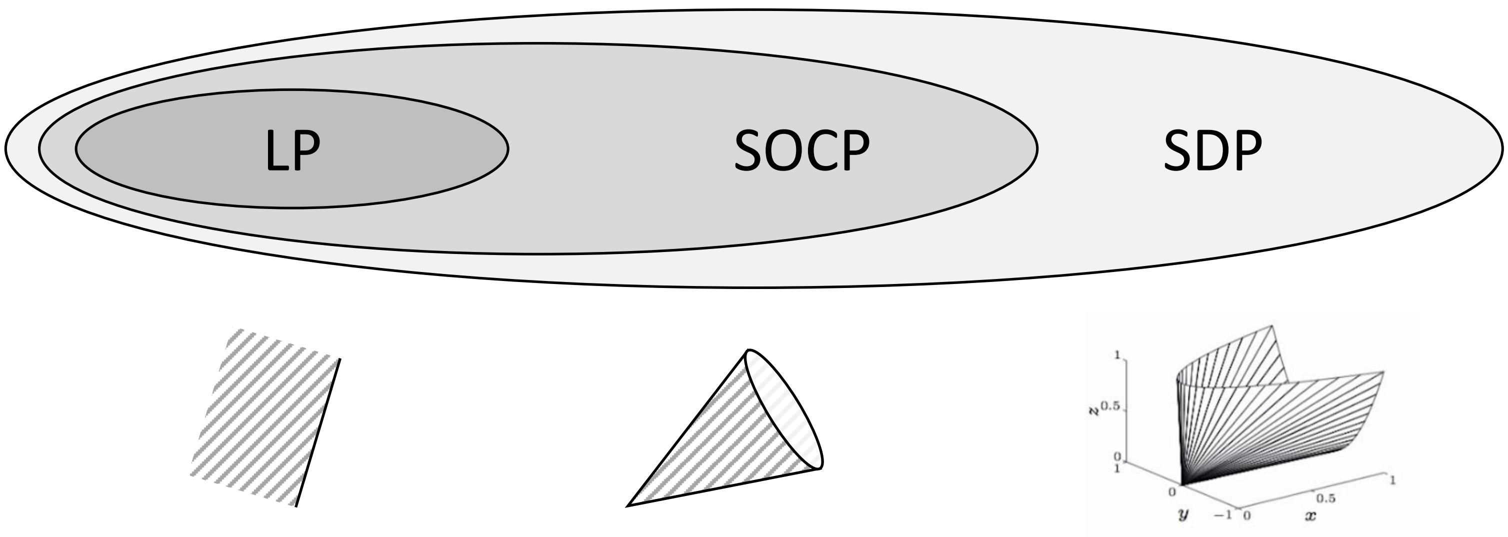
symmetric and $\lambda_i \geq 0$





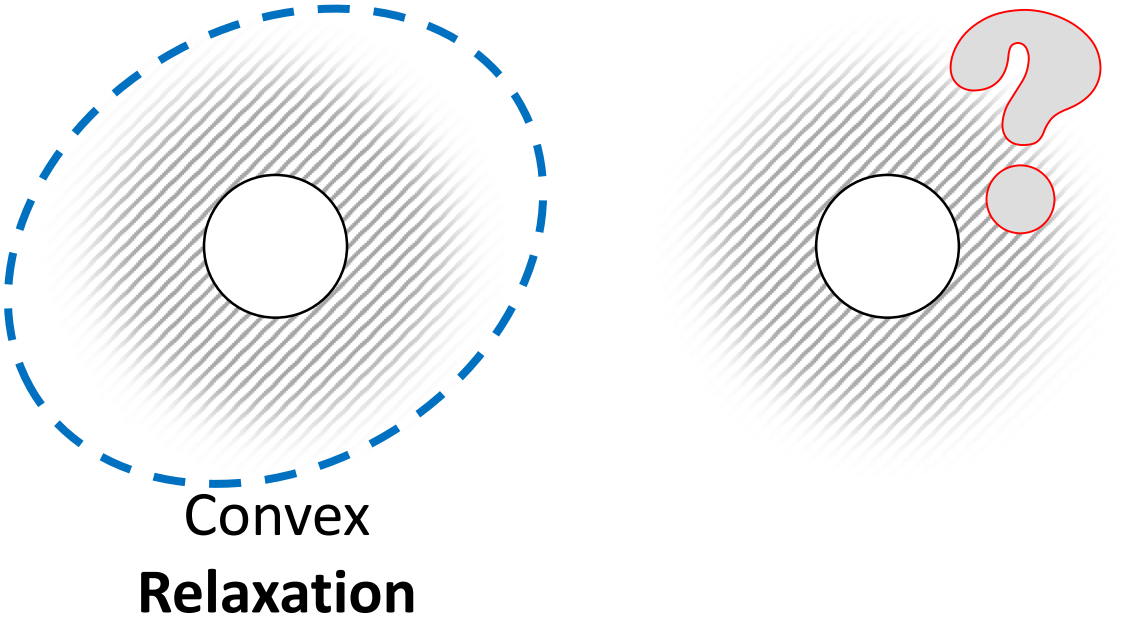

$\Rightarrow$ semidefinite programming (SDP)

# Hierarchy



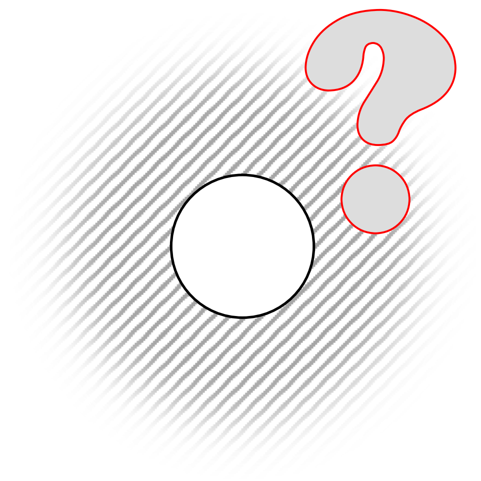

Guarantees & efficient optimization engines!

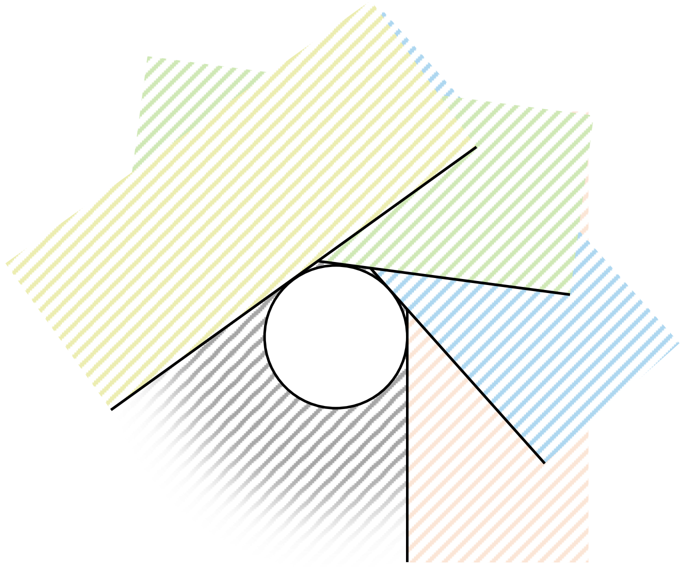
# Convexification



# Convexification

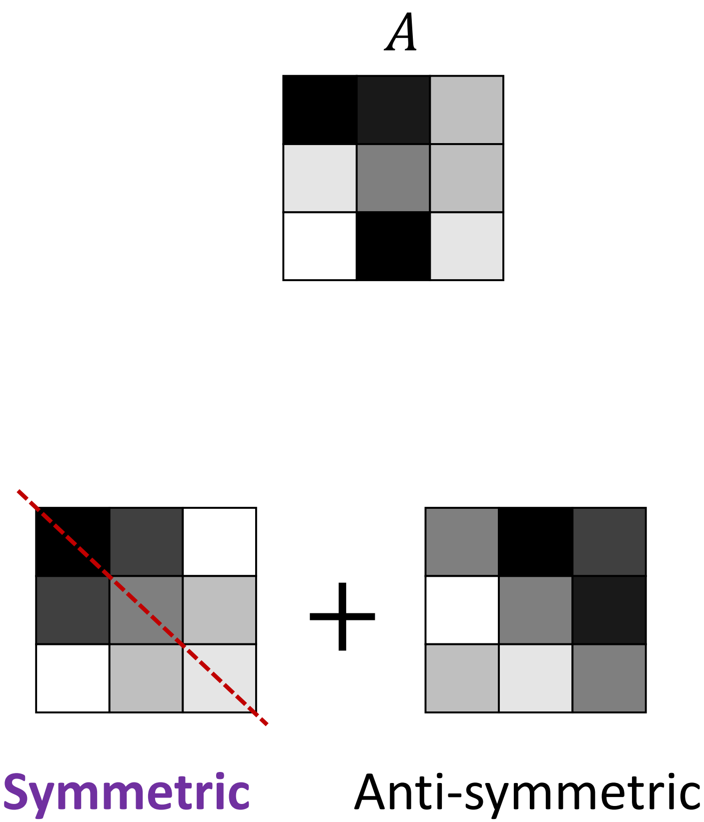
Convex
**Relaxation**





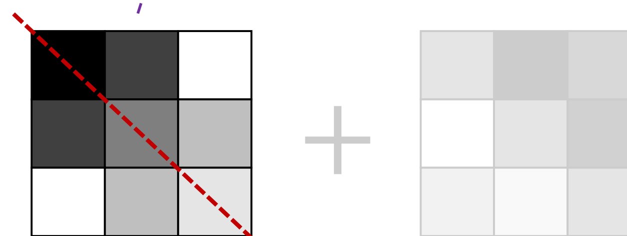
Convex
**Cover**

# Key observation



# Key observation

$$\boxed{\gamma \leq \sigma_{\min}(A)} \quad \Longleftarrow \quad \gamma \leq \sigma_{\min}\left(\frac{A + A^T}{2}\right)$$



**Symmetric** Anti-symmetric

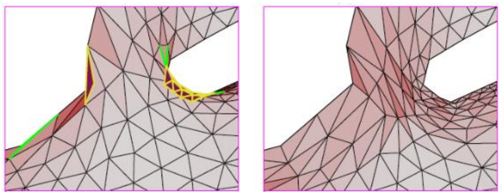
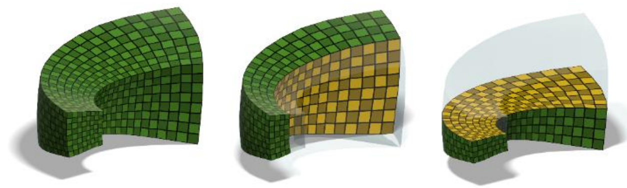
# Key observation

$\gamma \leq \sigma_{\min}(A)$ $\Leftarrow$ $\gamma \leq \sigma_{\min}\left(\frac{A + A^T}{2}\right)$

**SOCP** – 2d

**SDP** – 3d and higher

# 2d vs. 3d

| **2-d** | **3-d (and higher)** |
| --- | --- |
|  |  |
| | |
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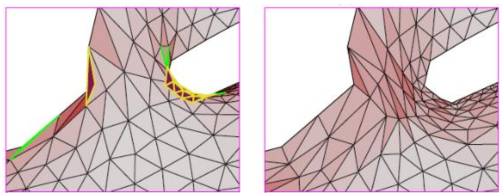
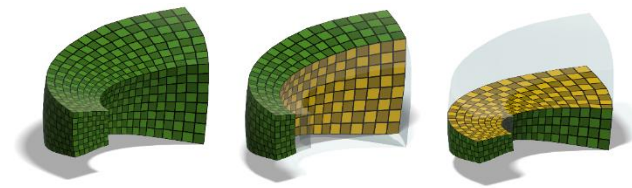
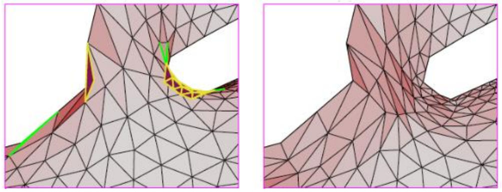
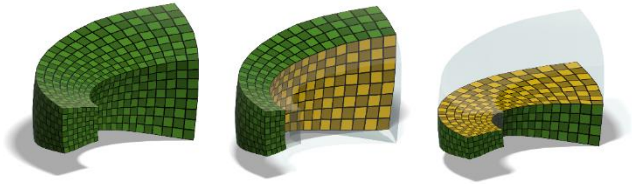
| | |
| | |

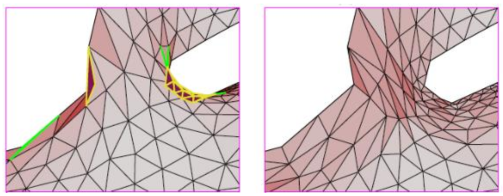
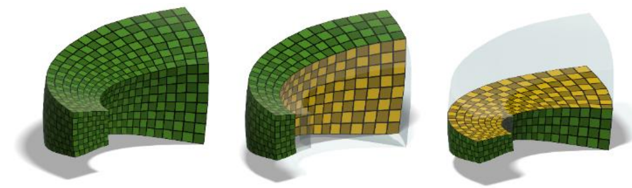
# 2d vs. 3d

| **2-d** | **3-d (and higher)** |
|---|---|
|  |  |
| $\mathcal{S}(A)$ has a closed **linear** form | $\mathcal{S}(A)$ is non-linear |
| | |
| | |
| | |

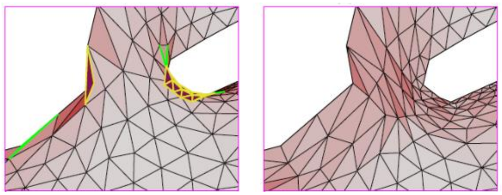
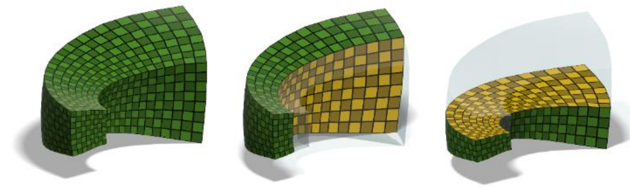
# 2d vs. 3d

| **2-d** | **3-d (and higher)** |
| --- | --- |
|  |  |
| $\mathcal{S}(A)$ has a closed **linear** form | $\mathcal{S}(A)$ is non-linear |
| $R_1 R_2 = R_2 R_1$ | $R_1 R_2 \neq R_2 R_1$ |
| | |

# 2d vs. 3d

| **2-d** | **3-d (and higher)** |
|---|---|
|  |  |
| $\mathcal{S}(A)$ has a closed **linear** form | $\mathcal{S}(A)$ is non-linear |
| $R_1 R_2 = R_2 R_1$ | $R_1 R_2 \neq R_2 R_1$ |
| $\sigma_i$'s have a closed form | roots of $\geq 6^{\text{th}}$ degree poly |
| | |

# 2d vs. 3d

| **2-d** | **3-d (and higher)** |
|---|---|
|  |  |
| $\mathcal{S}(A)$ has a closed **linear** form | $\mathcal{S}(A)$ is non-linear |
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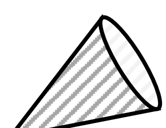
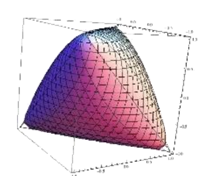
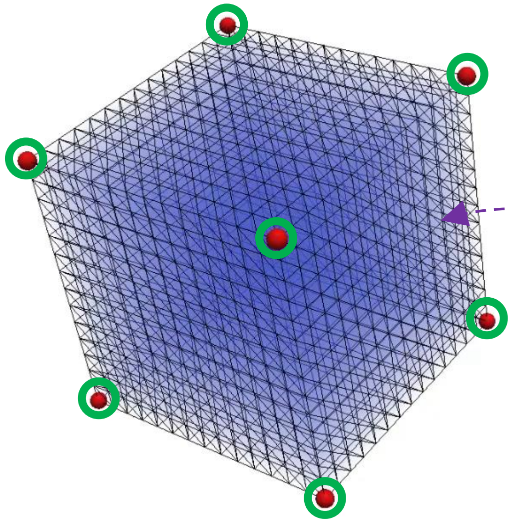
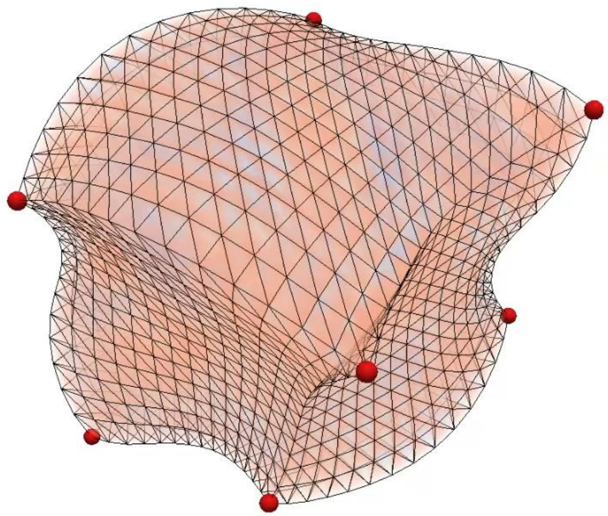
| **SOCP**  | **SDP**  |

# Extremal Quasiconformal Mappings



$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

# Extremal Quasiconformal Mappings

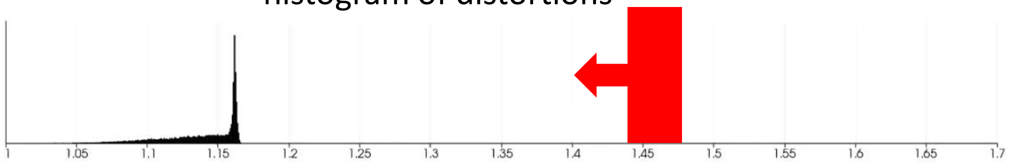


$$\textbf{minimize}\left(\max_i \frac{\sigma_{\max}(A_i)}{\sigma_{\min}(A_i)}\right)$$

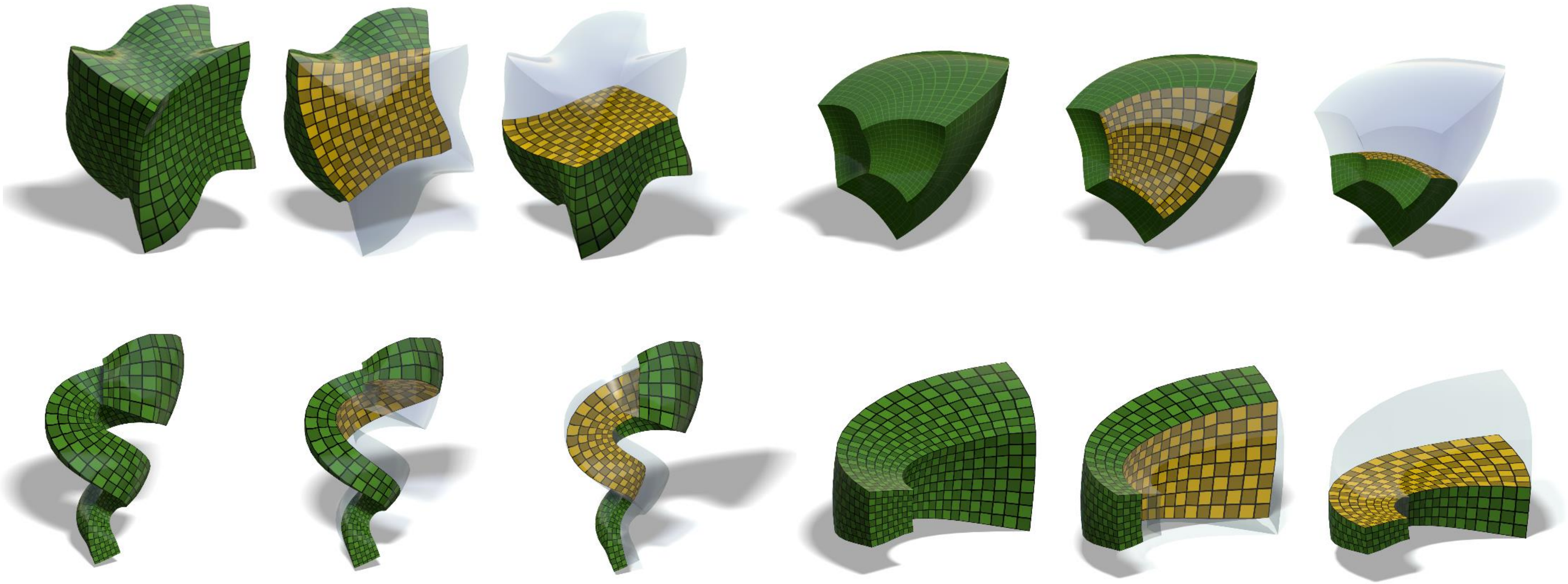
**"Most Conformal Mapping"**

- Well studied in 2D [Weber et al. 2012]
- Little known in 3D…

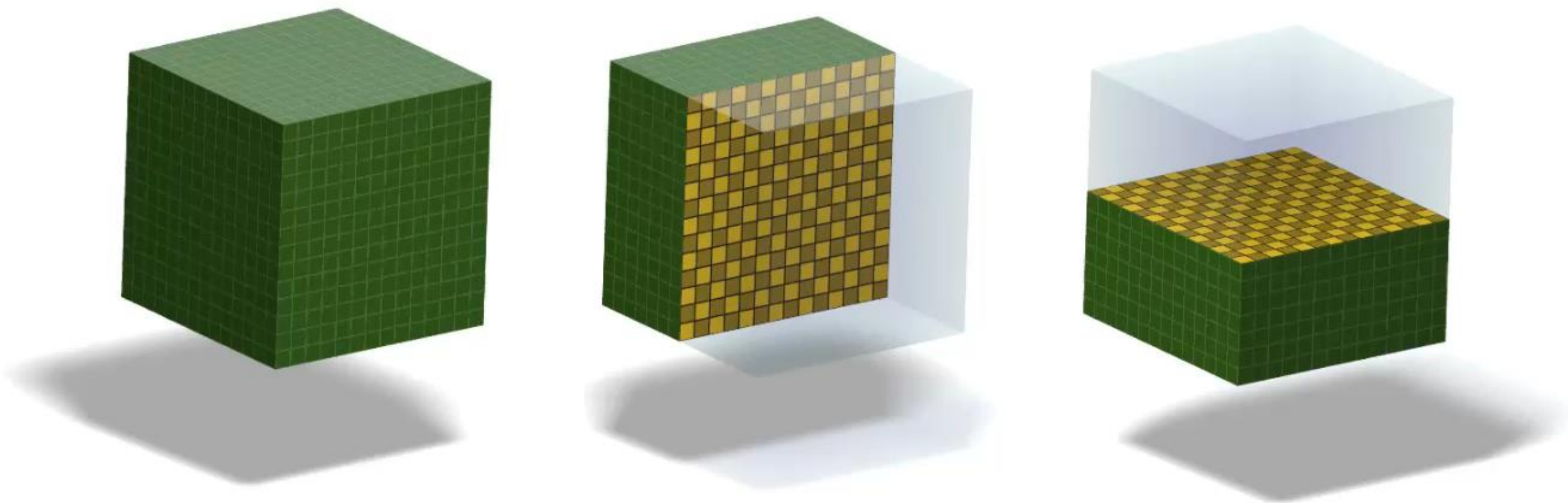
histogram of distortions



# Extremal Quasiconformal Mappings



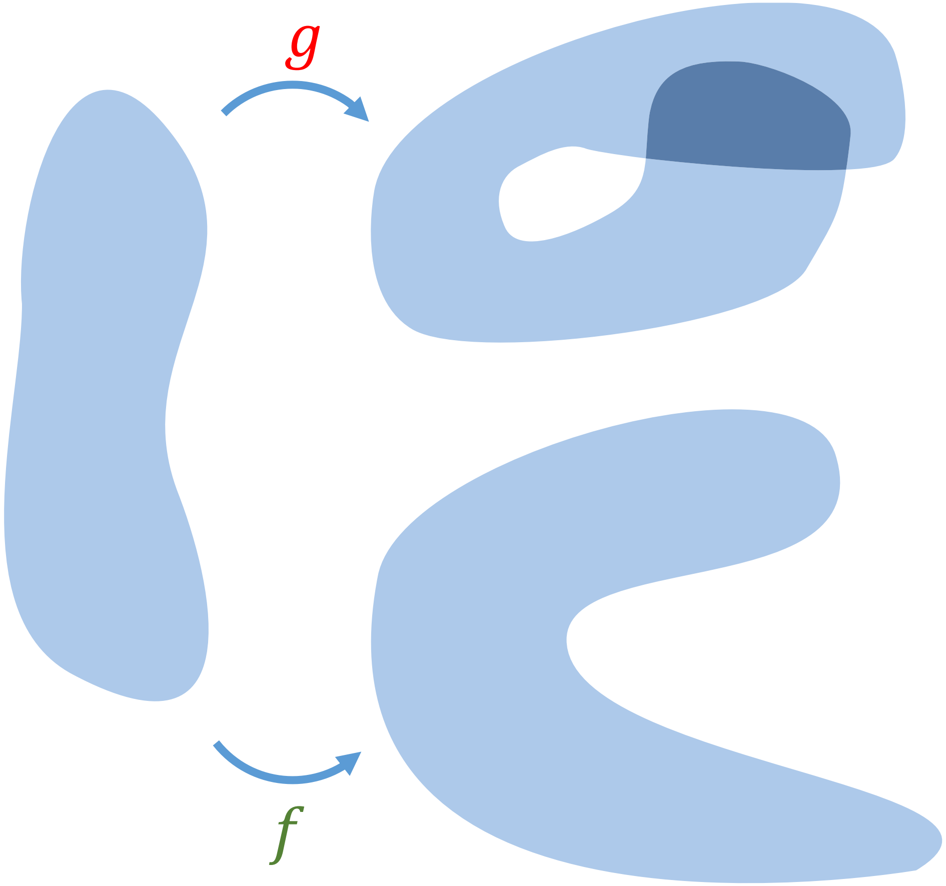
# Extremal Quasiconformal Mappings



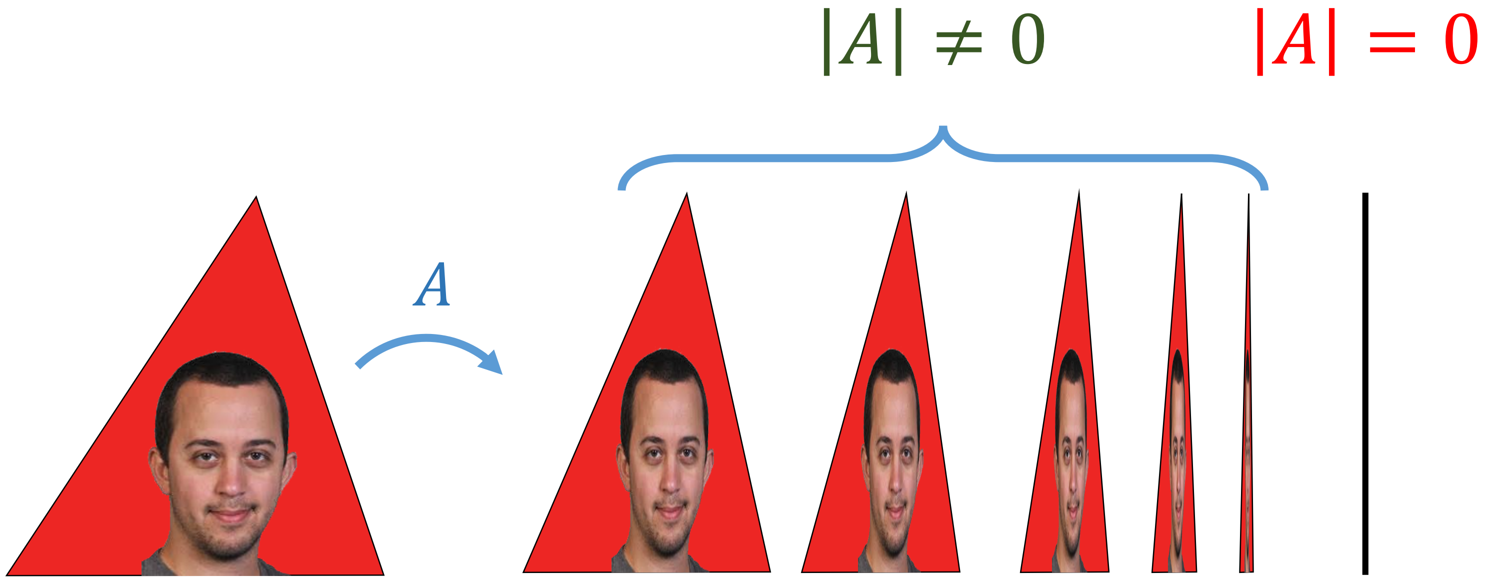
# Injectivity

# Injectivity

- “Map is 1-to-1”

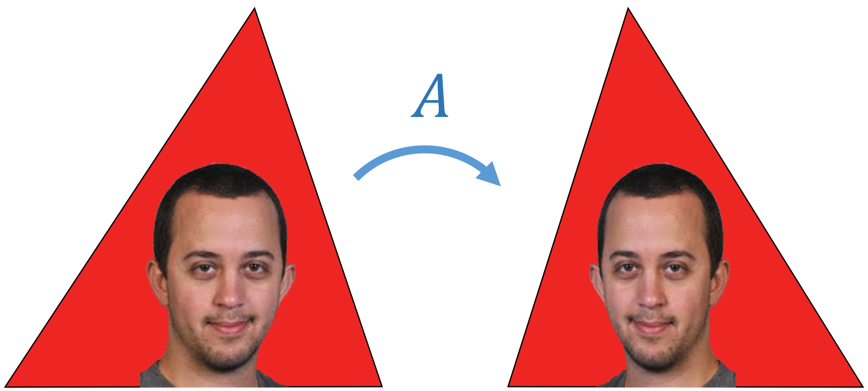


# Injective affine map



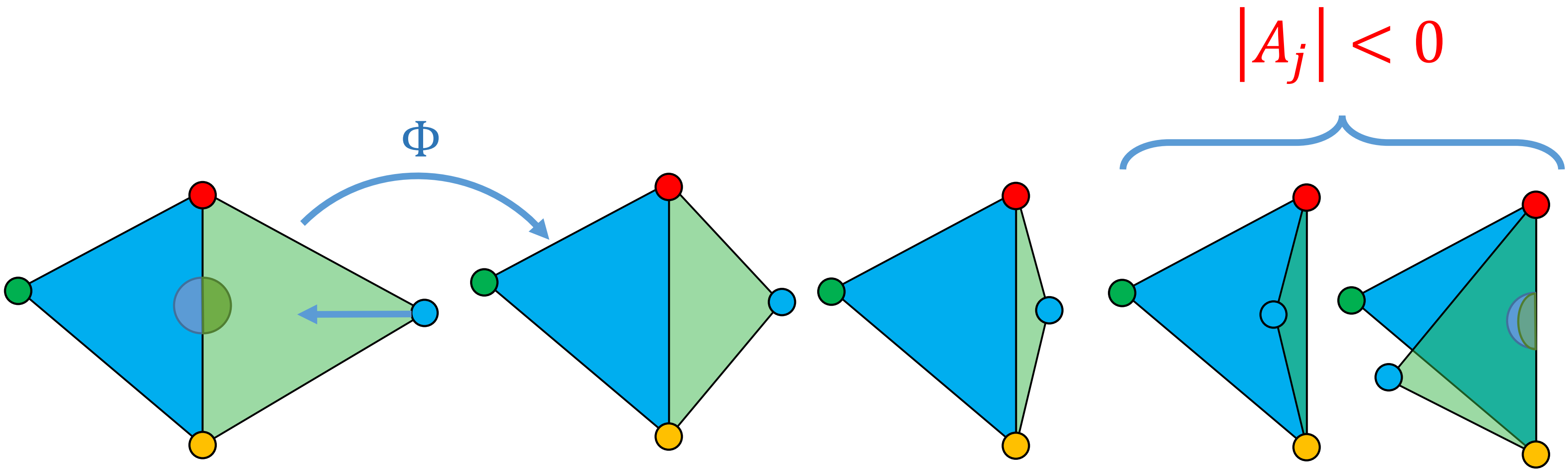
# Injective affine map

$$|A| < 0$$



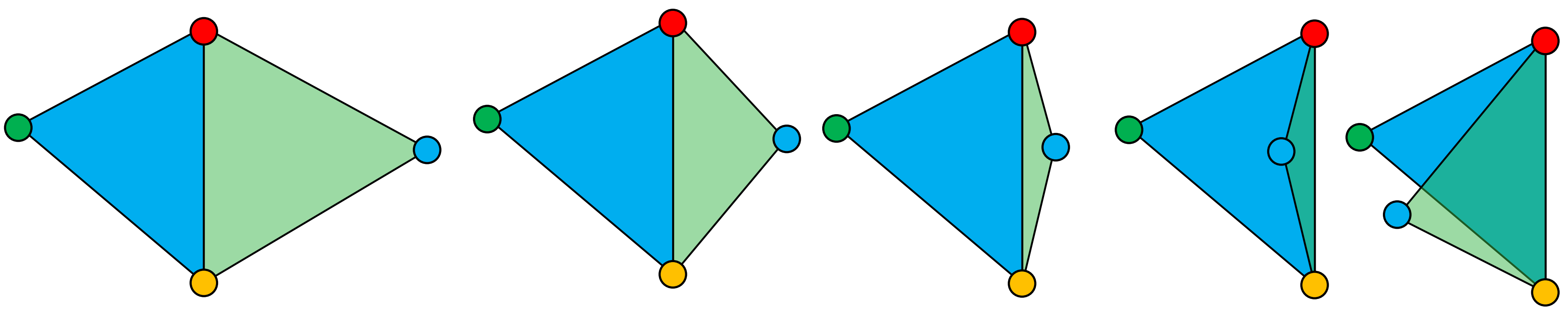
# Injective **piecewise** affine map

- Different orientation?
- Not injective on edges



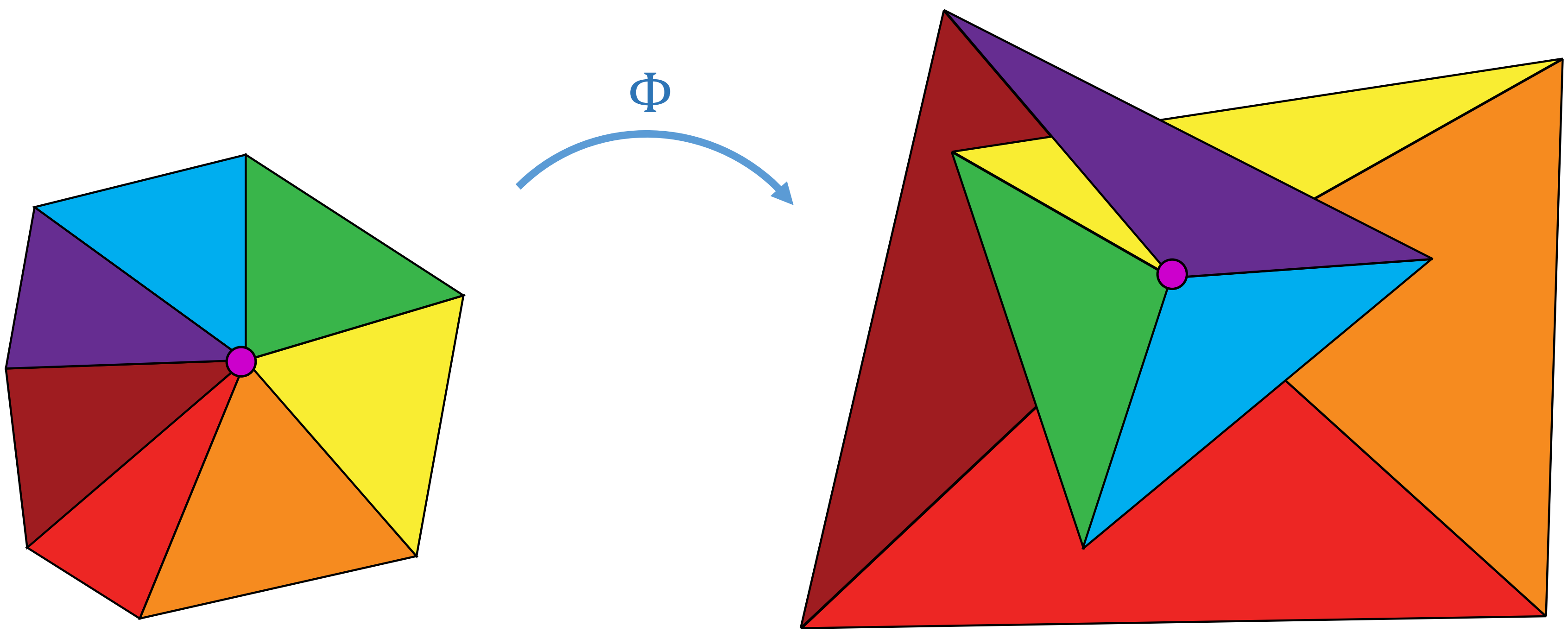
# Injective piecewise affine map

- Injecitivity requires consistent orientation!
  - Is it sufficient?



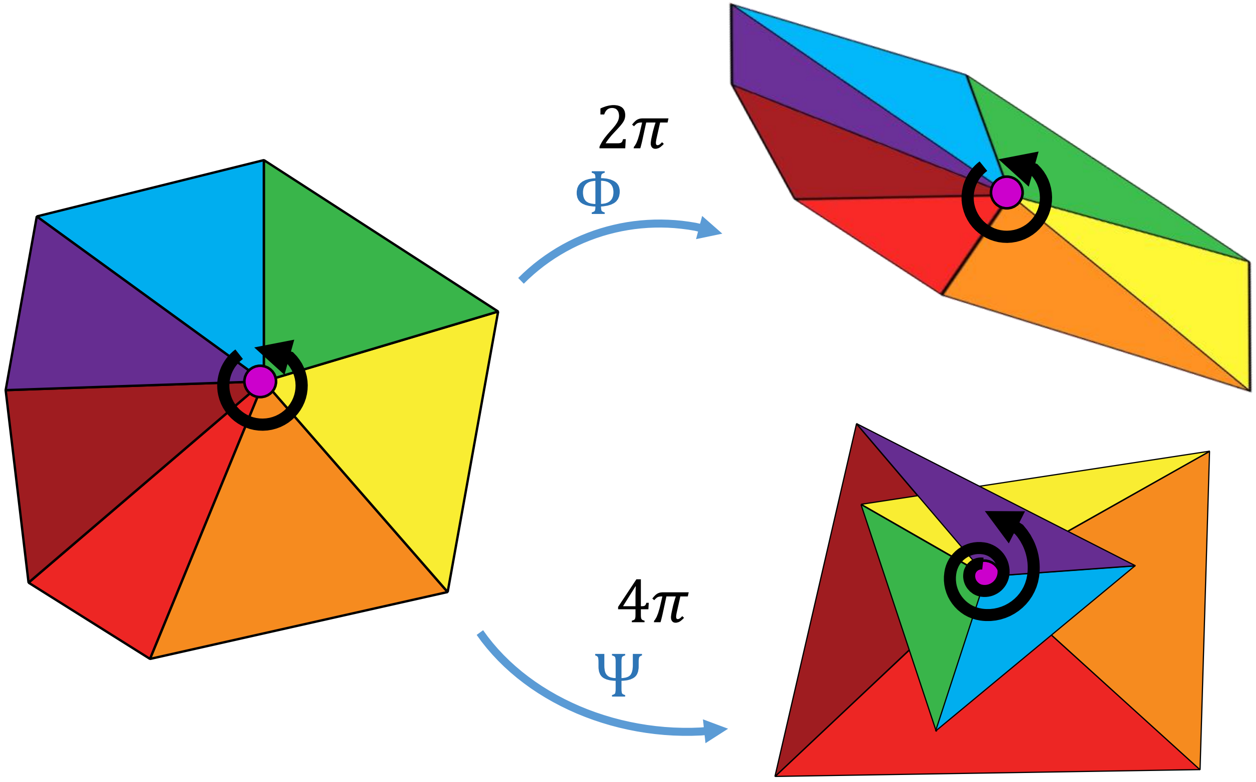
# Injective piecewise affine map

- Consistent orientation
- Not injective only on inner vertex



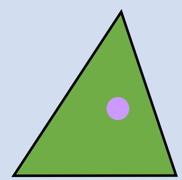
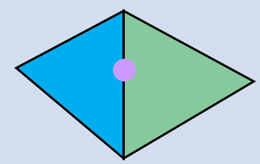
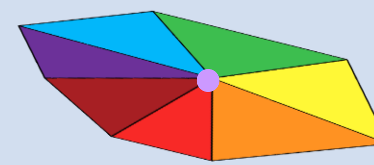
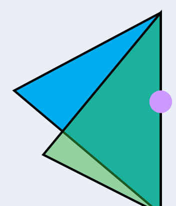
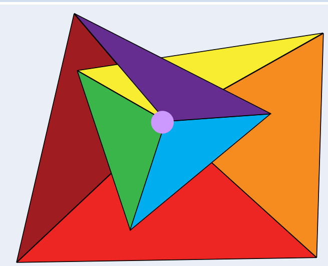
# Injective piecewise affine map

- Winding around vertex should be $2\pi$



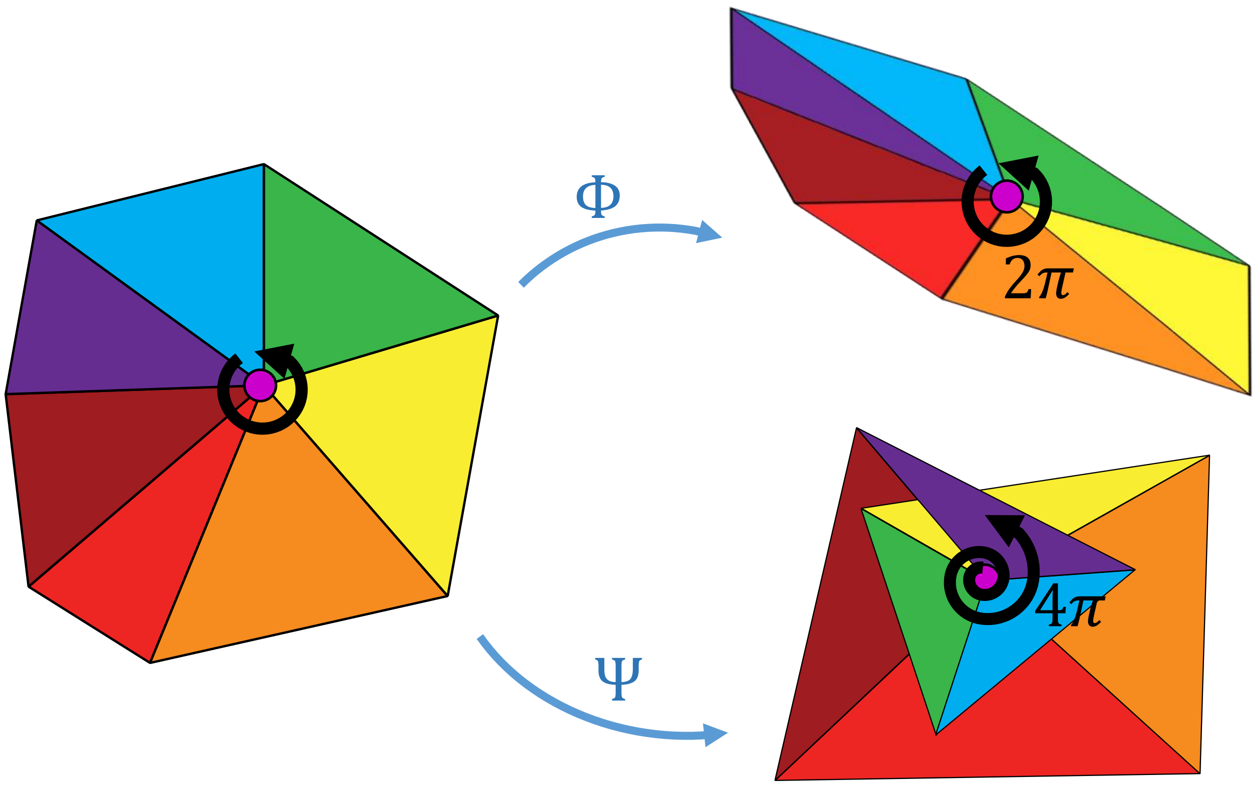
# Local Injectivity

- "in a small neighborhood, we are injective"

| | Face | Edge | vertex |
|---|---|---|---|
| Locally Injective |  |  |  |
| Non-Injective |  |  |  |
| | $\|A\| \neq 0$ | $\|A\| > 0$ | $\sum\theta_i = 2\pi$ |

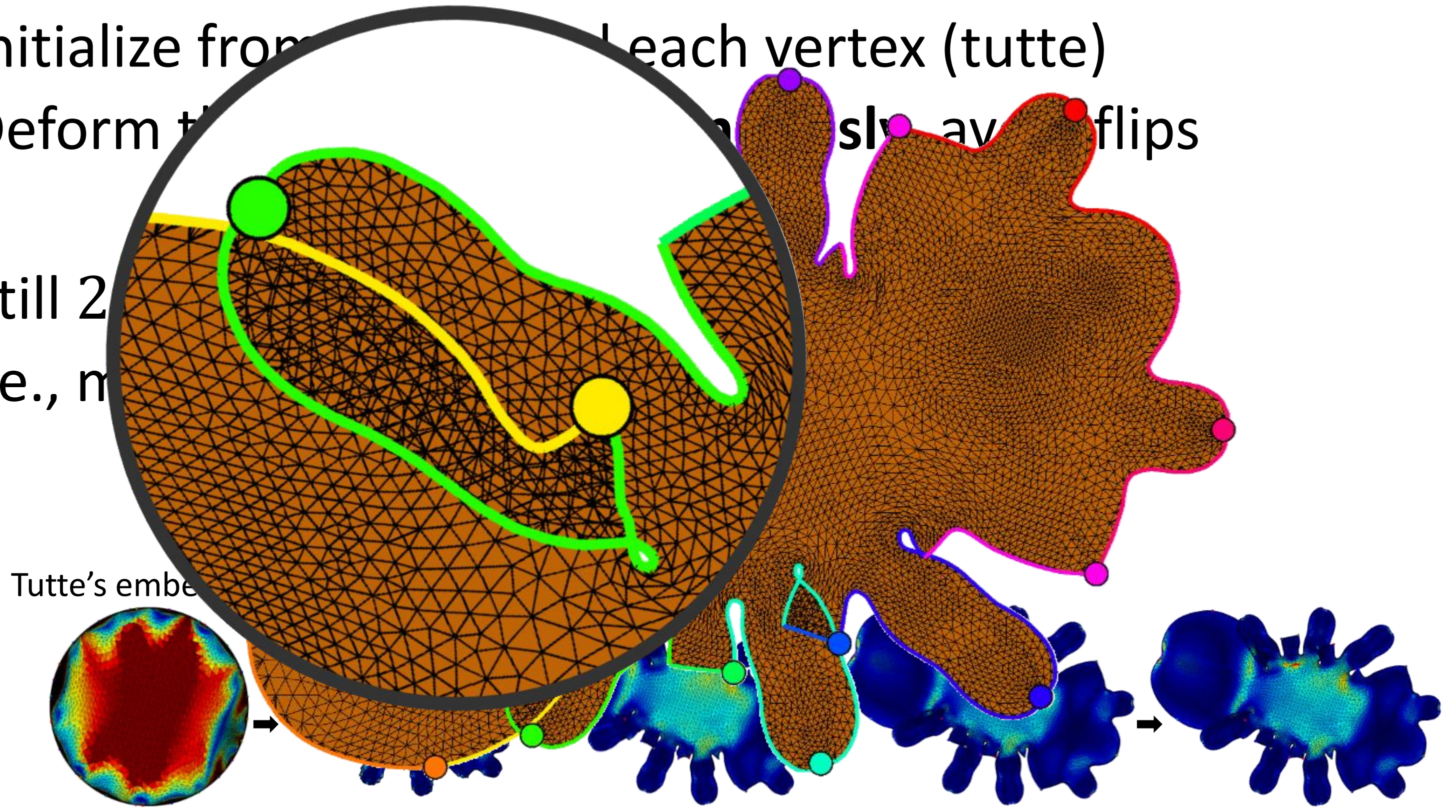
# Injective piecewise affine map

- Note: winding around vertex is always $2\pi k$



# Injective piecewise affine map

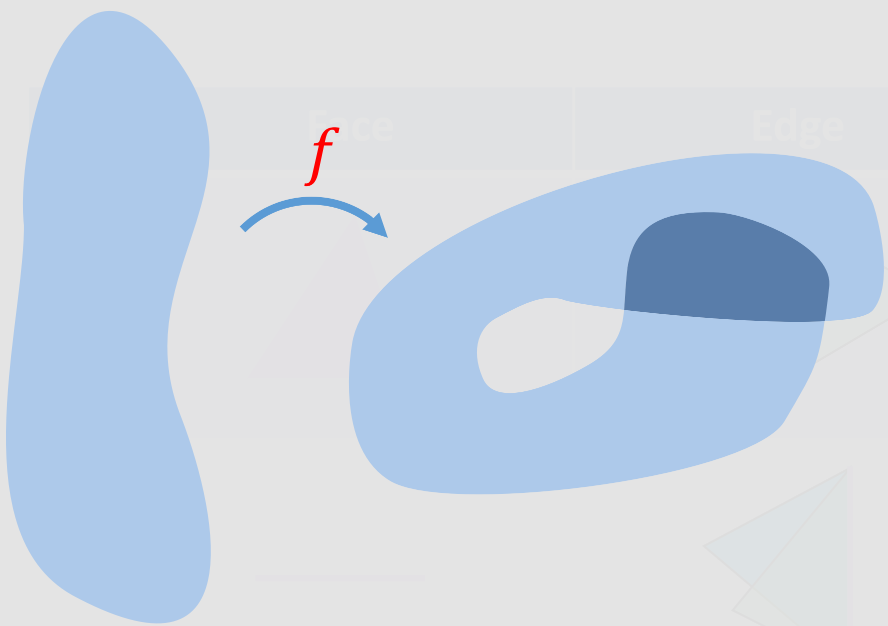
- if:
  - Initialize fro[m] … each vertex (tutte)
  - Deform t… sl… av… flips
- Then:
  - Still 2…
  - i.e., m…




[Aigerman2014]

# Local Injectivity

- “in small neighborhood, we are injective”

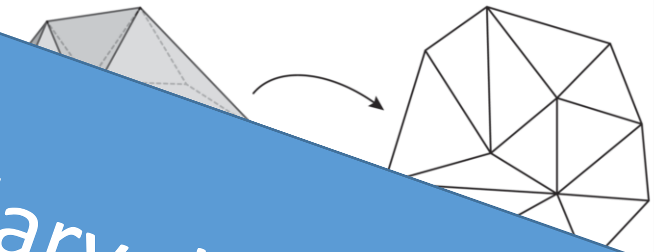



- Local inj < Global inj
- e.g., $f$ is locally inj

**Important on its own!**

# Global injectivity?

- [Tut 1961]: my embedding is injective!

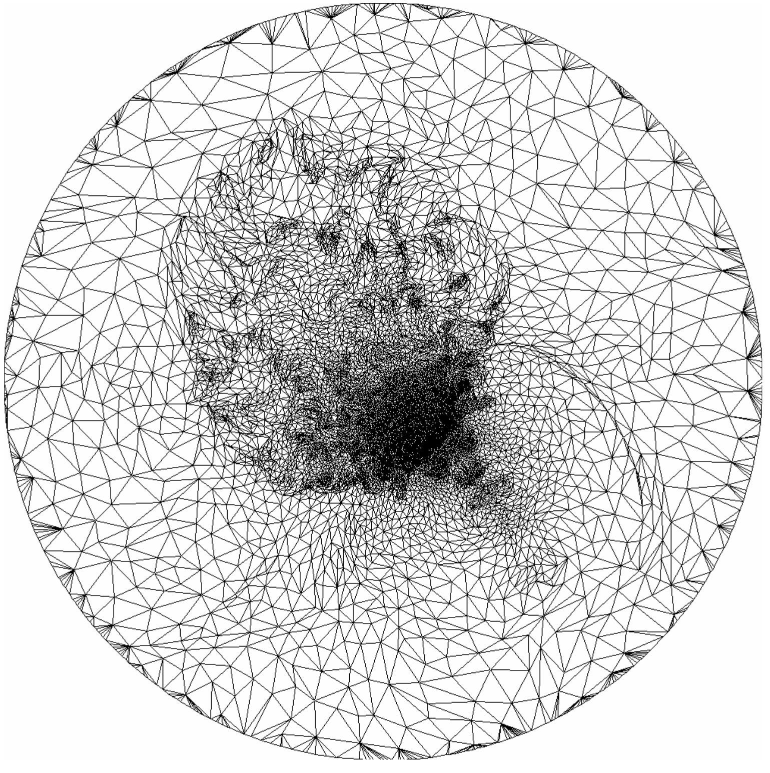


Boundary doesn’t overlap?

- [Lipman201 ap is
  injective + no flips,

# Injective boundary

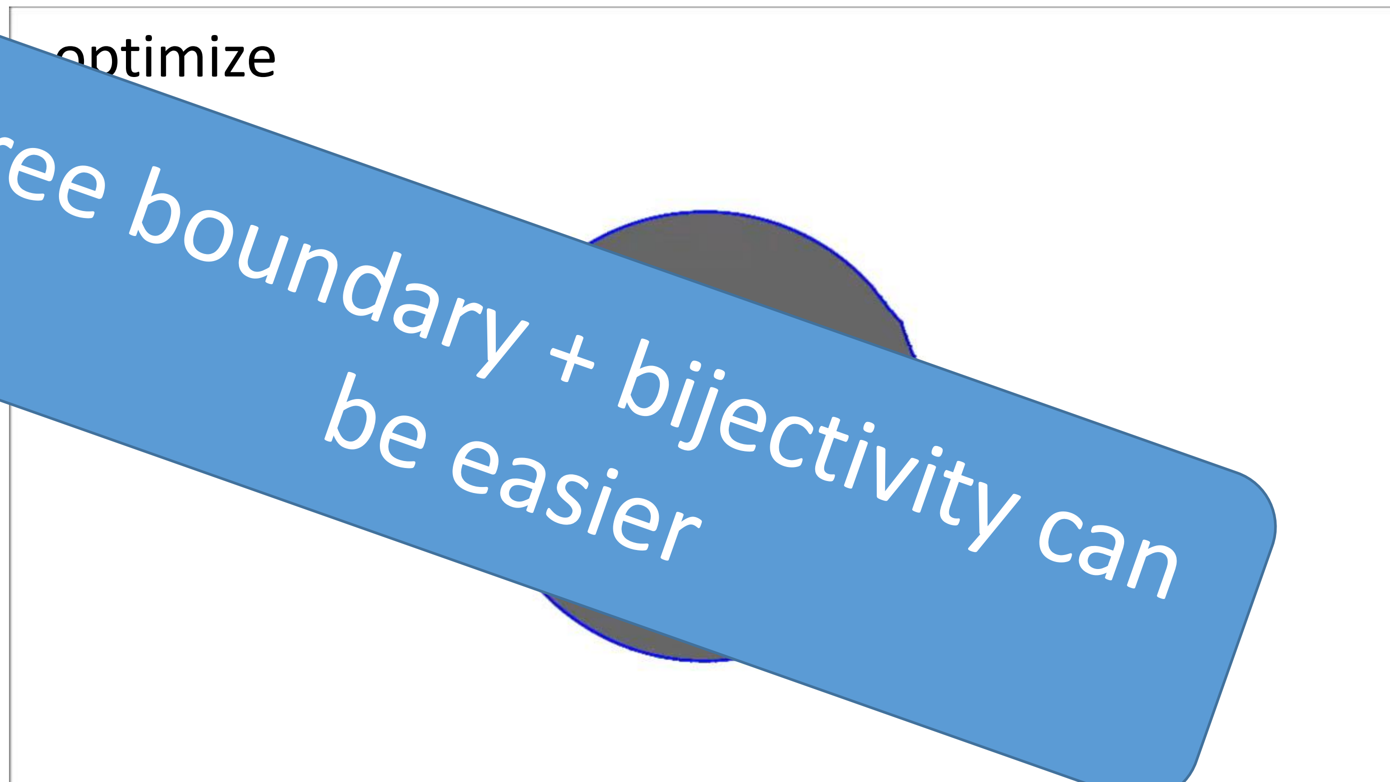
- Fixed
  - Highly constrained



[Gortler et. al 2006]

# Injective boundary

- Prevent boundary from overlapping during optimization
  - Ha optimize

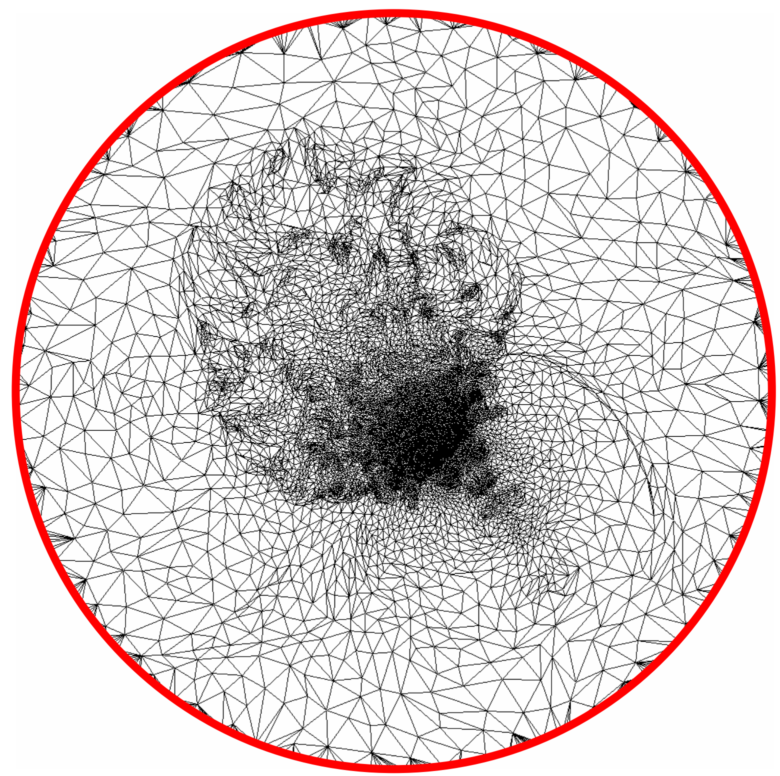


Free boundary + bijectivity can be easier

[Smith and Schaefer 2015]

# Variations on a theme

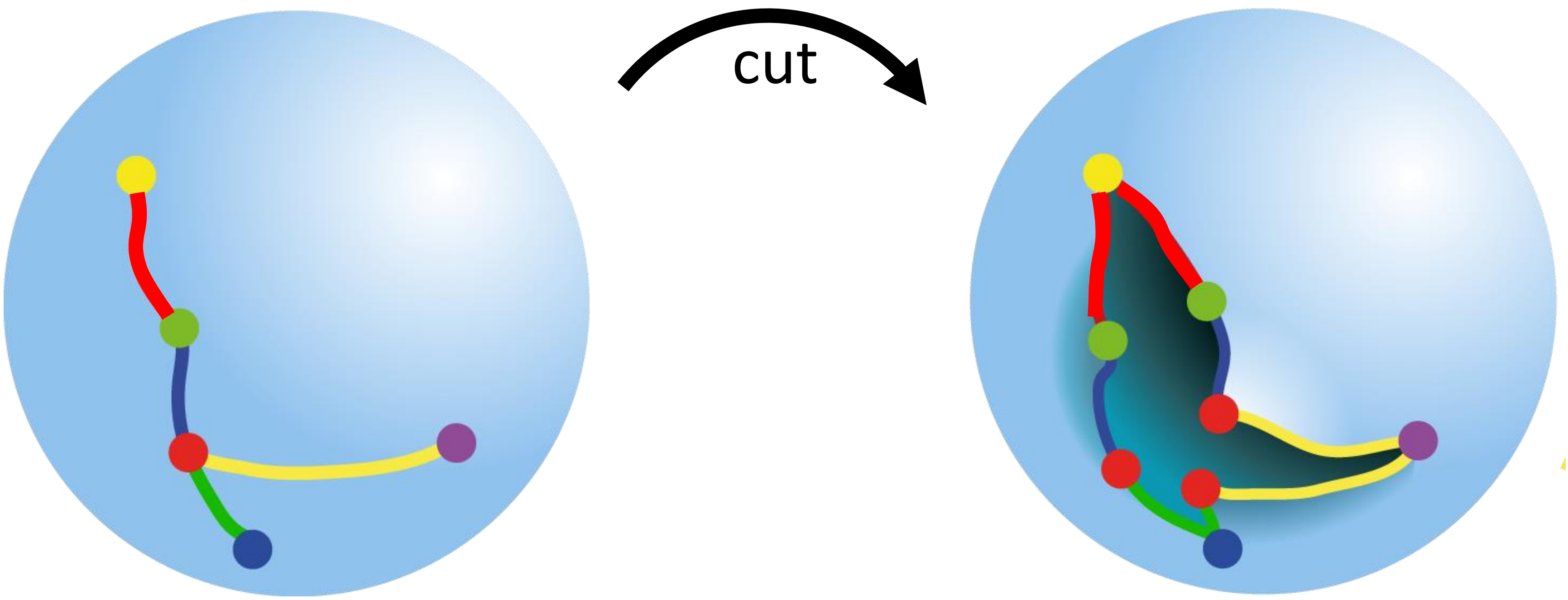
# Parameterizing a disk…



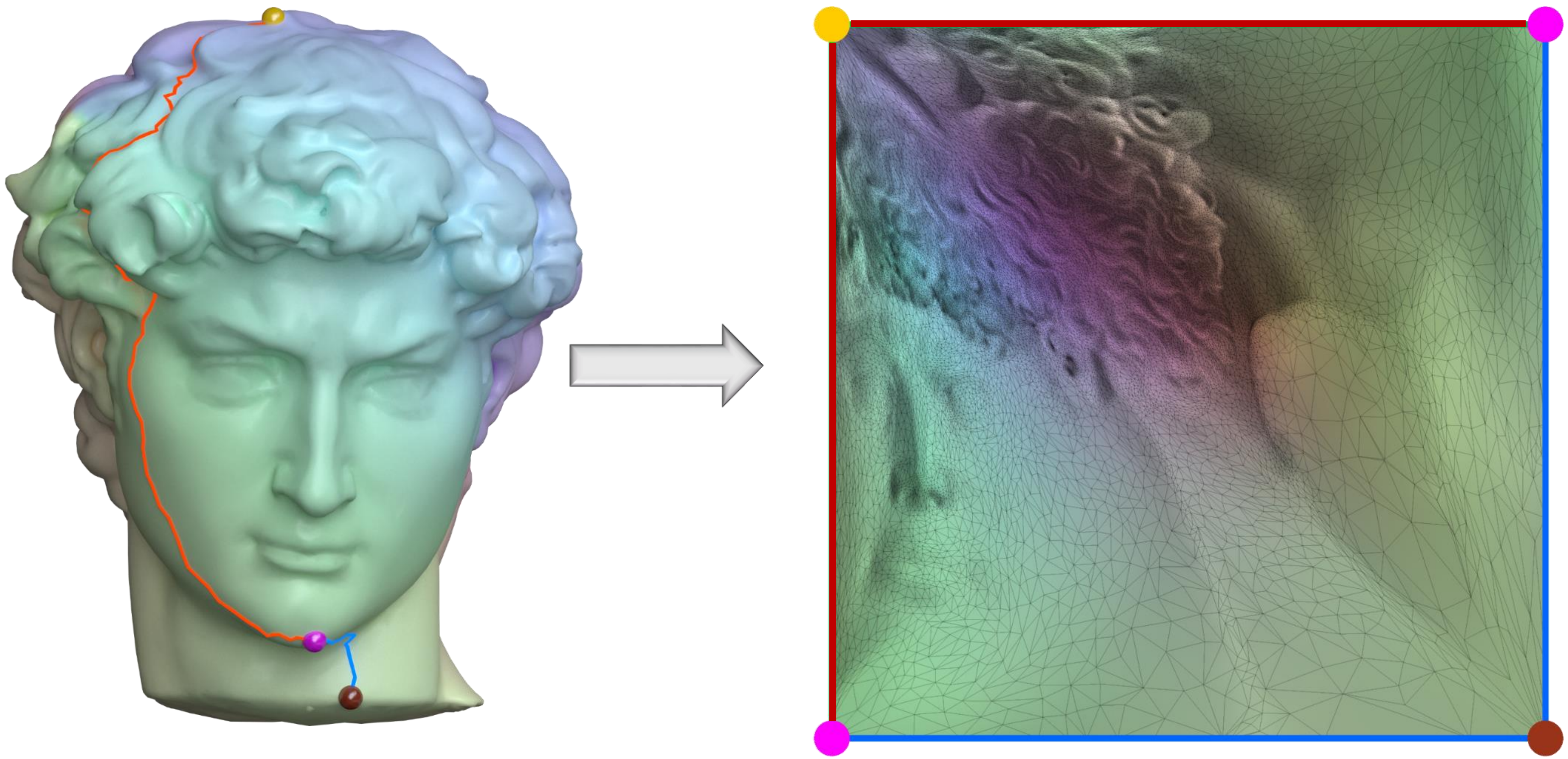
# Parameterizing a sphere?

# Naïve solution for spheres

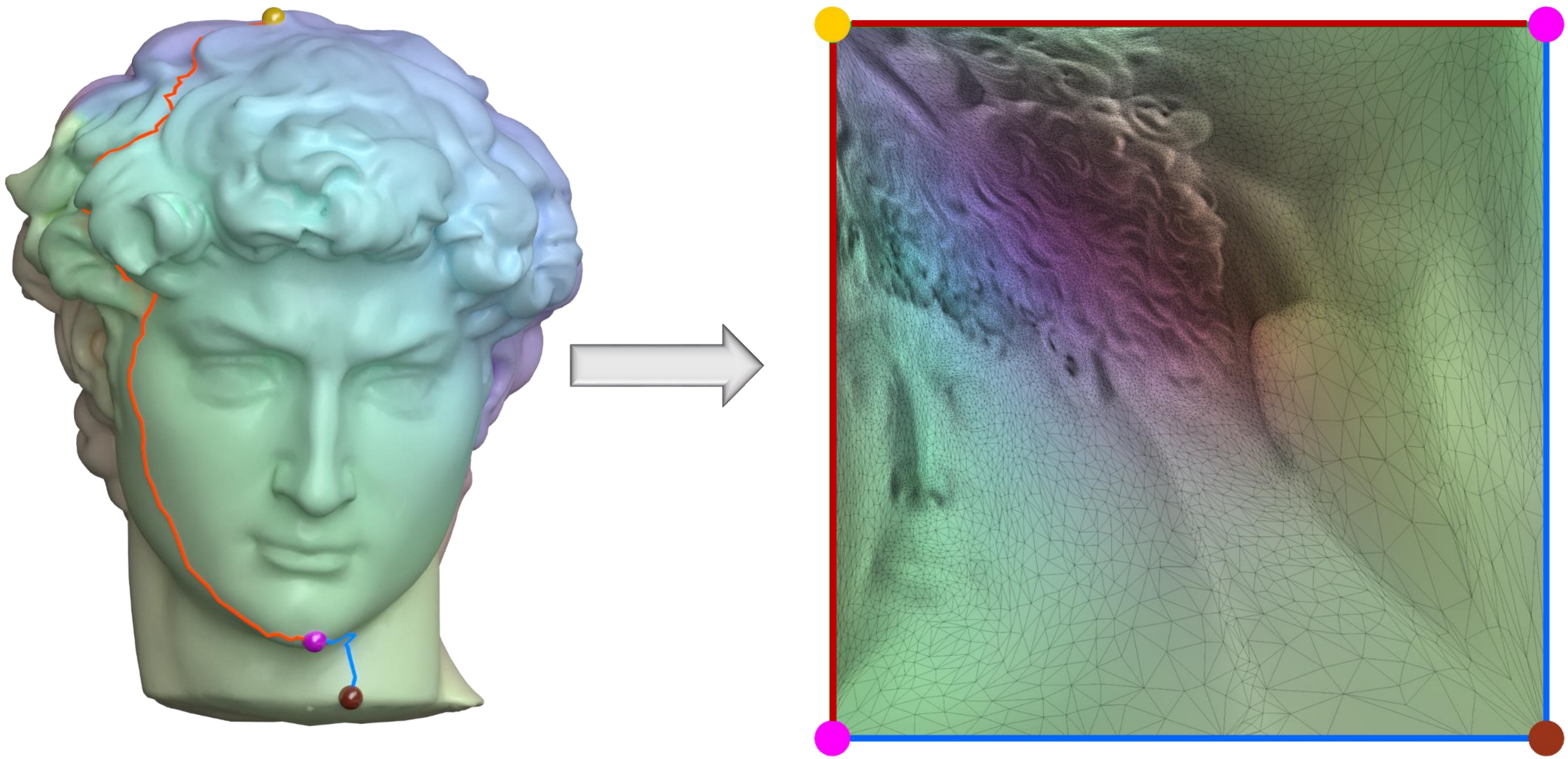
Reduce spheres to disks…



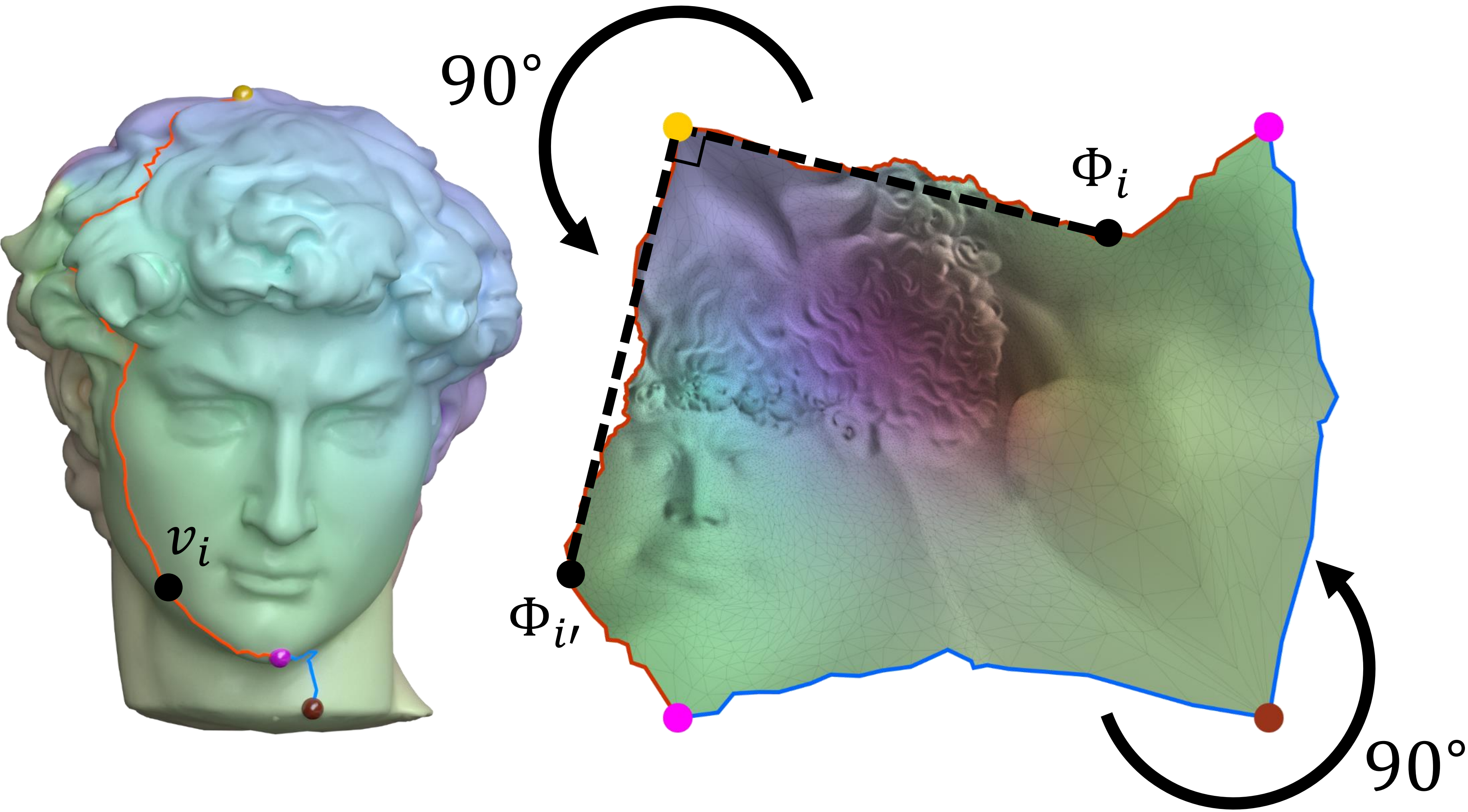
# Kind of unnatural



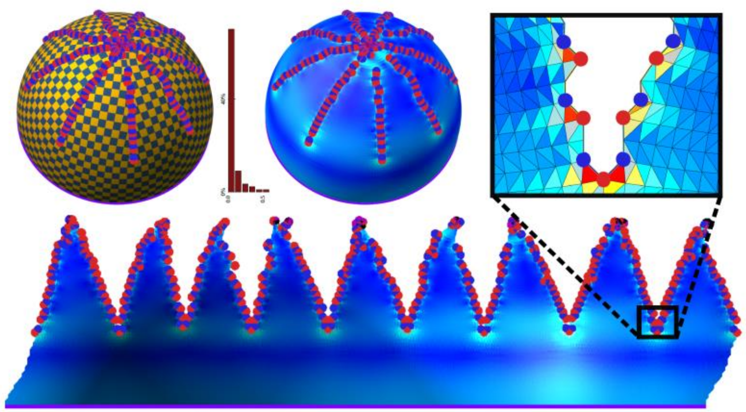
# Instead of fixed boundary...



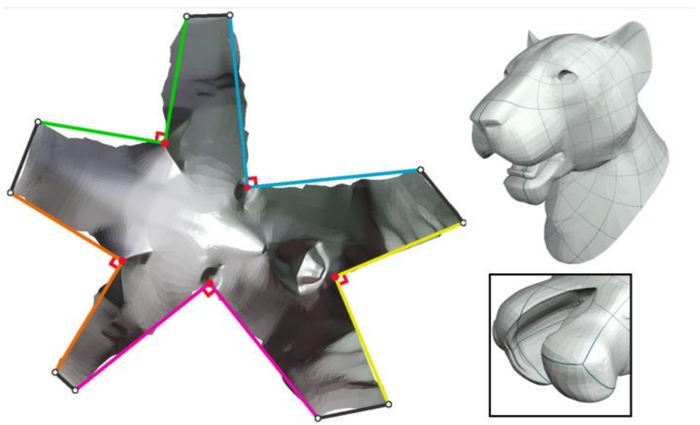
Periodic boundary!



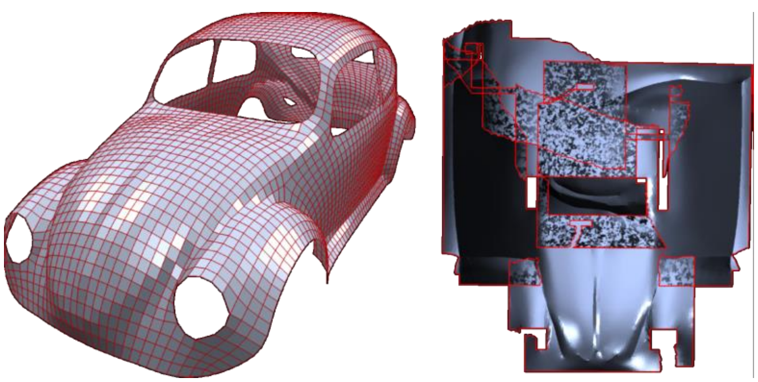
# Cone manifolds



[Myles and Zorin 2013]



[Springborn *et al.* 2008]

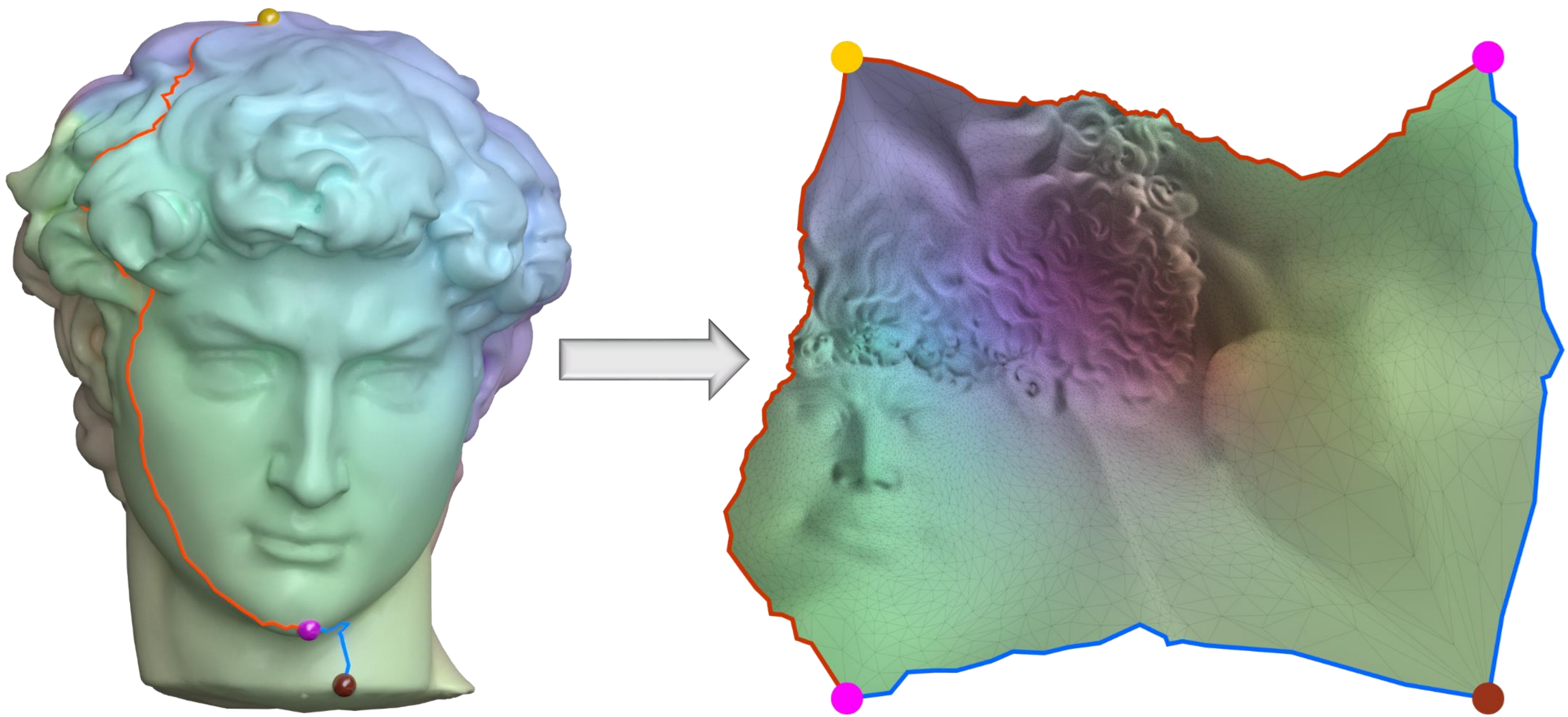


[Bommes *et al.* 2009]

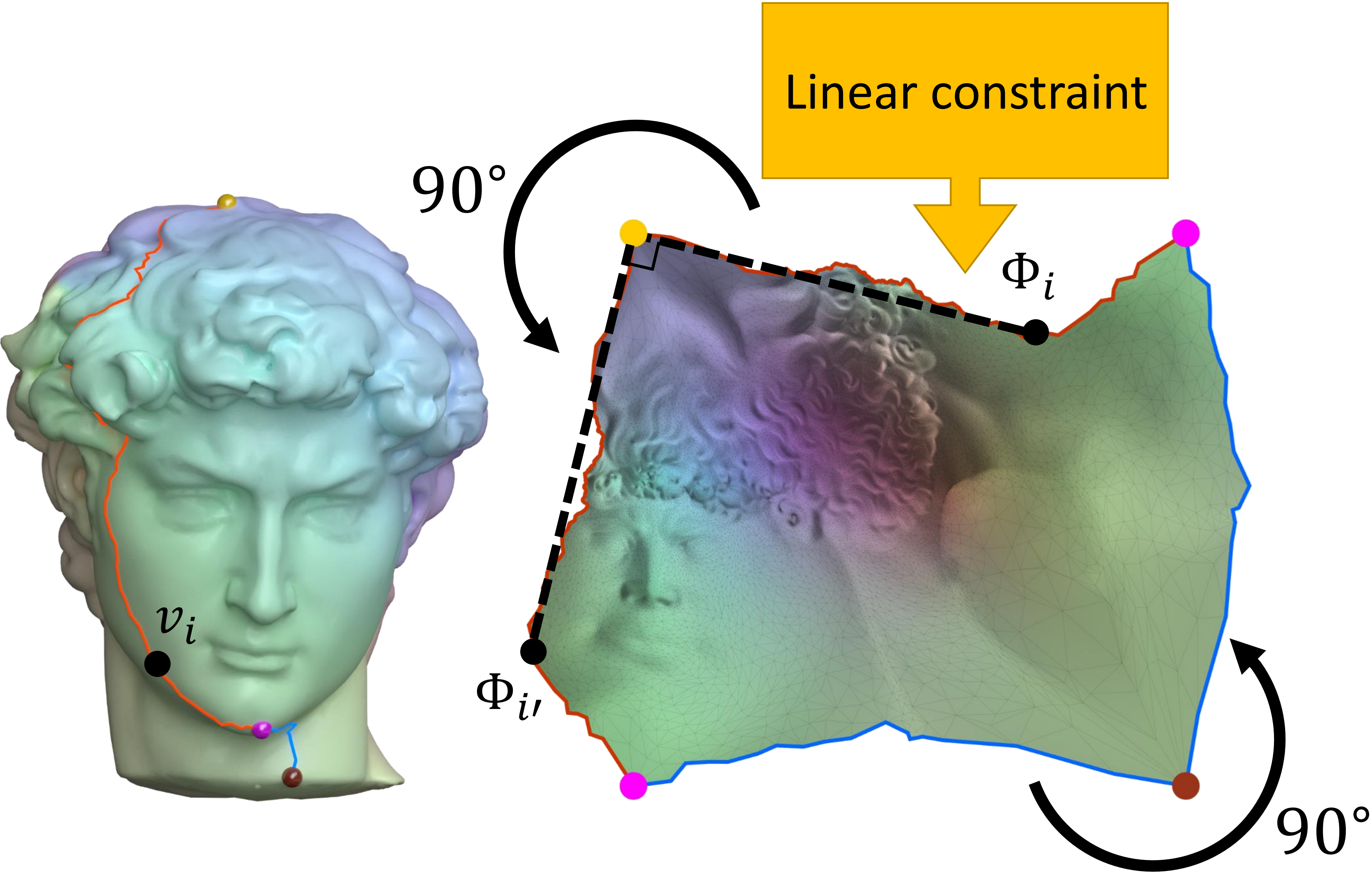
[Aigerman and Lipman 2015]

# Tutte for sphere

## Linear conditions for bijective parameterization

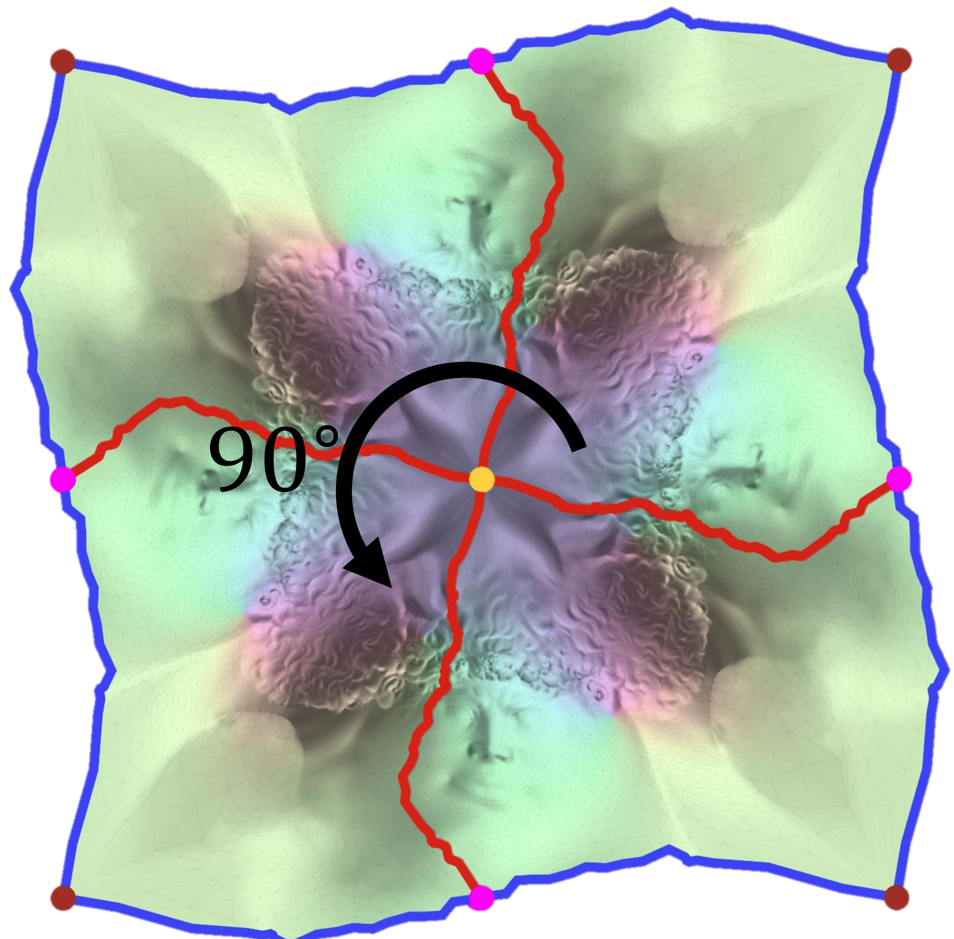


# 1. Periodic Boundary



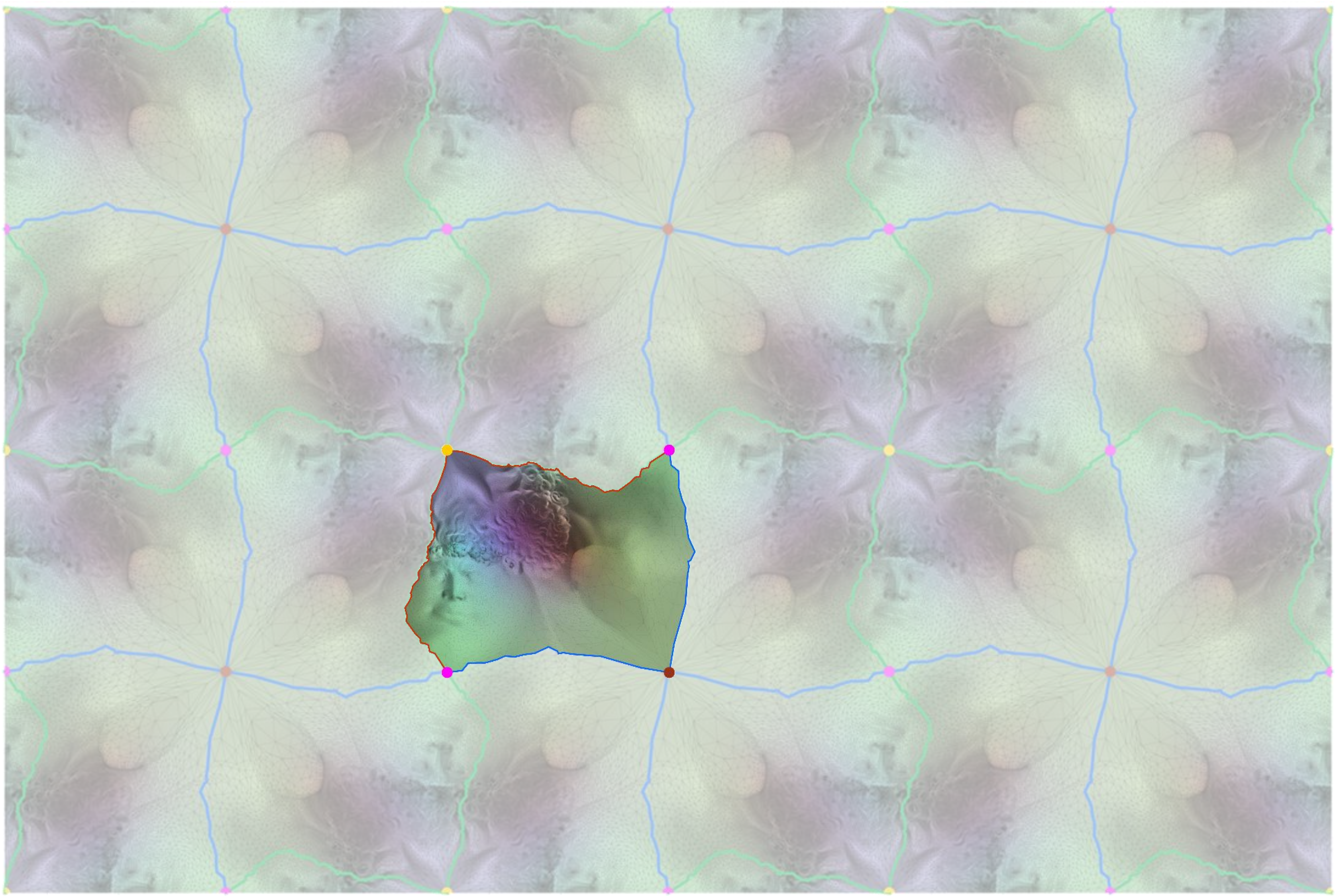
# 1. Periodic Boundary

Can glue copies across cuts!



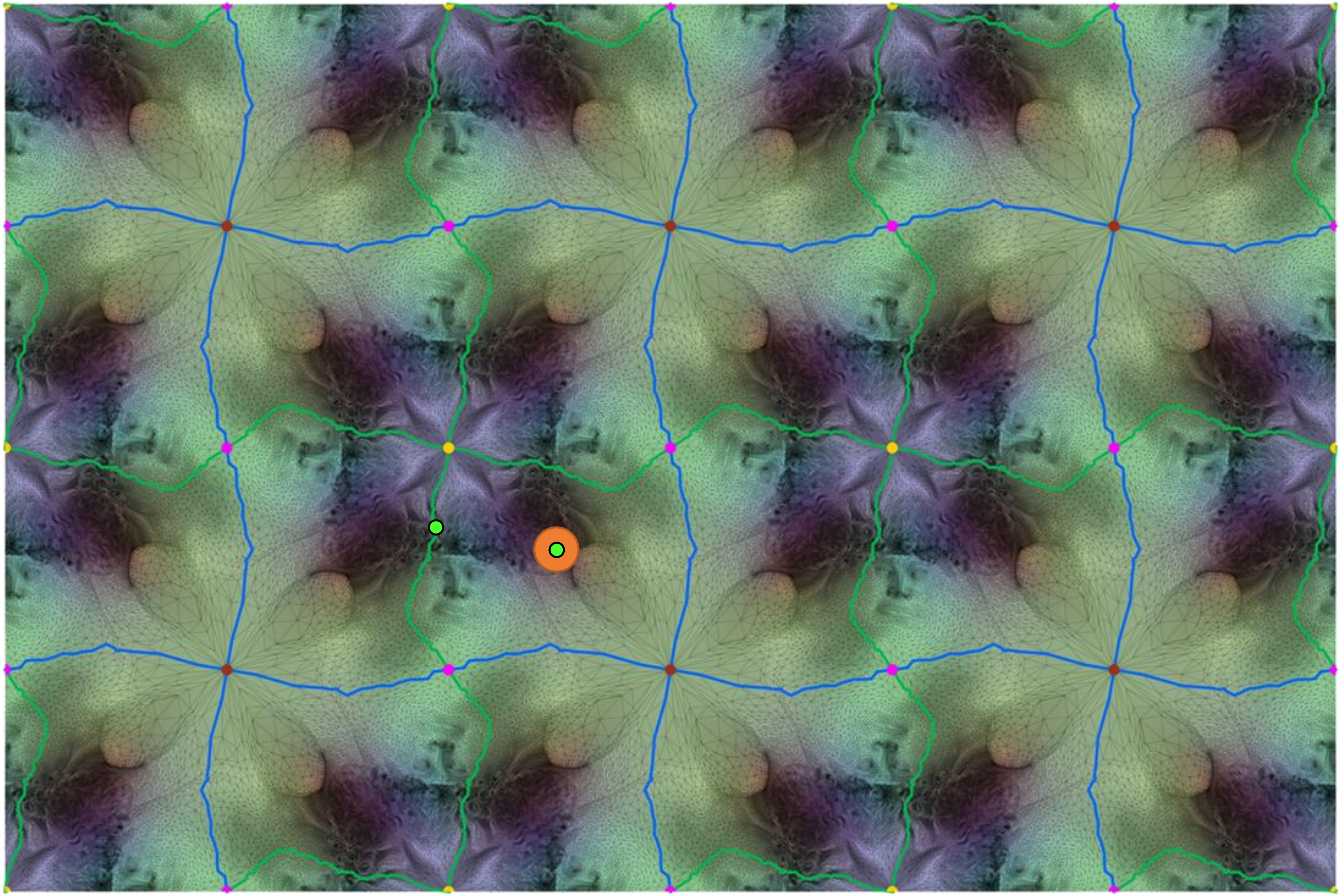
# 1. Periodic Boundary

Can (conceptually) tile the whole plane

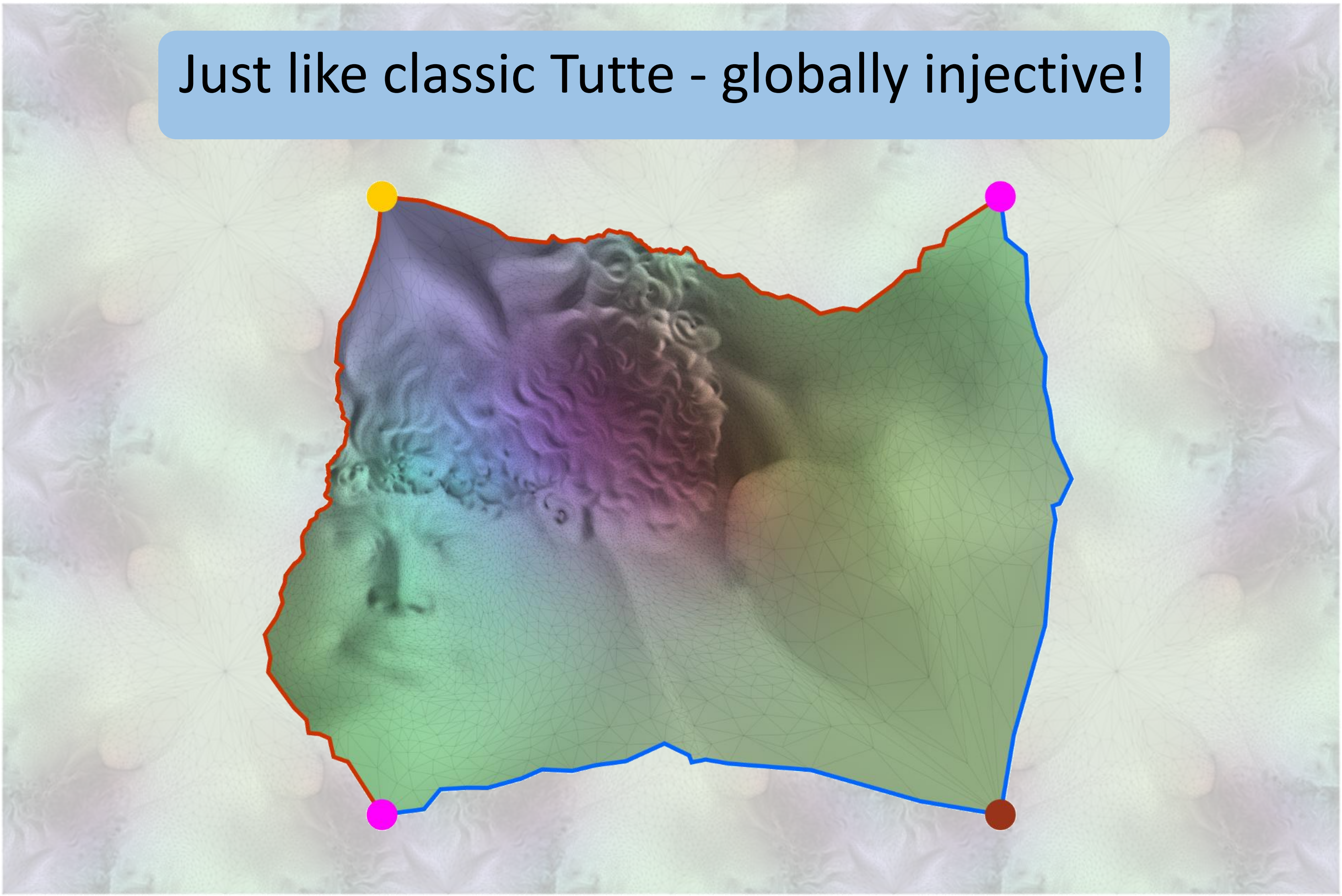


# 2. Discrete Harmonic Tiling

Each vertex in average of neighbours



# Harmonic tiling of $\mathbb{R}^2$



# Orbifold Tutte Embeddings

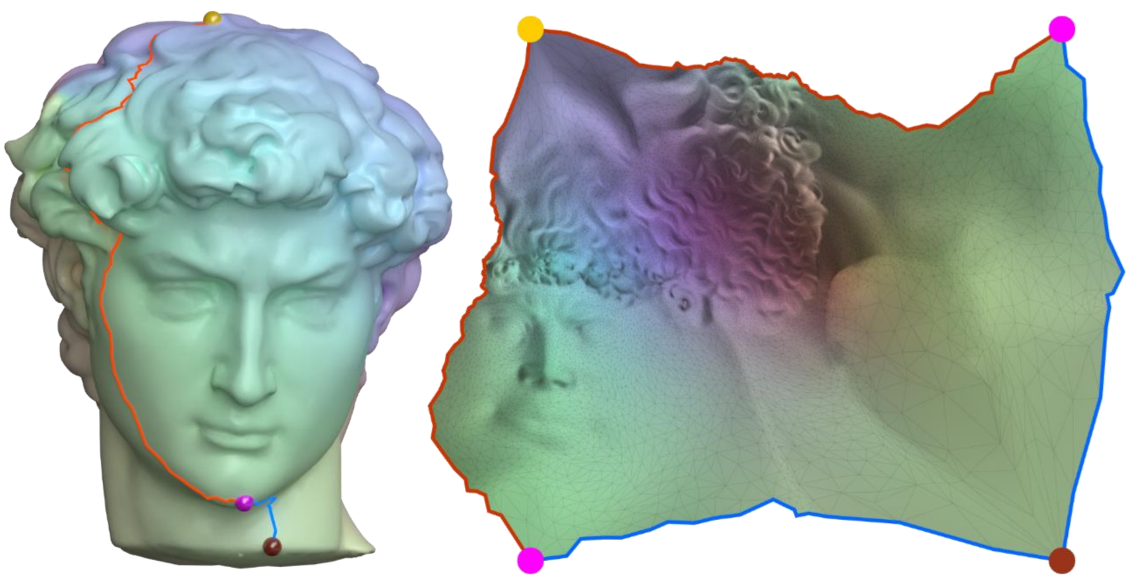
If:

1. Boundaries are rotated copies that tile the plane
2. The tiling is harmonic **everywhere**

Then:

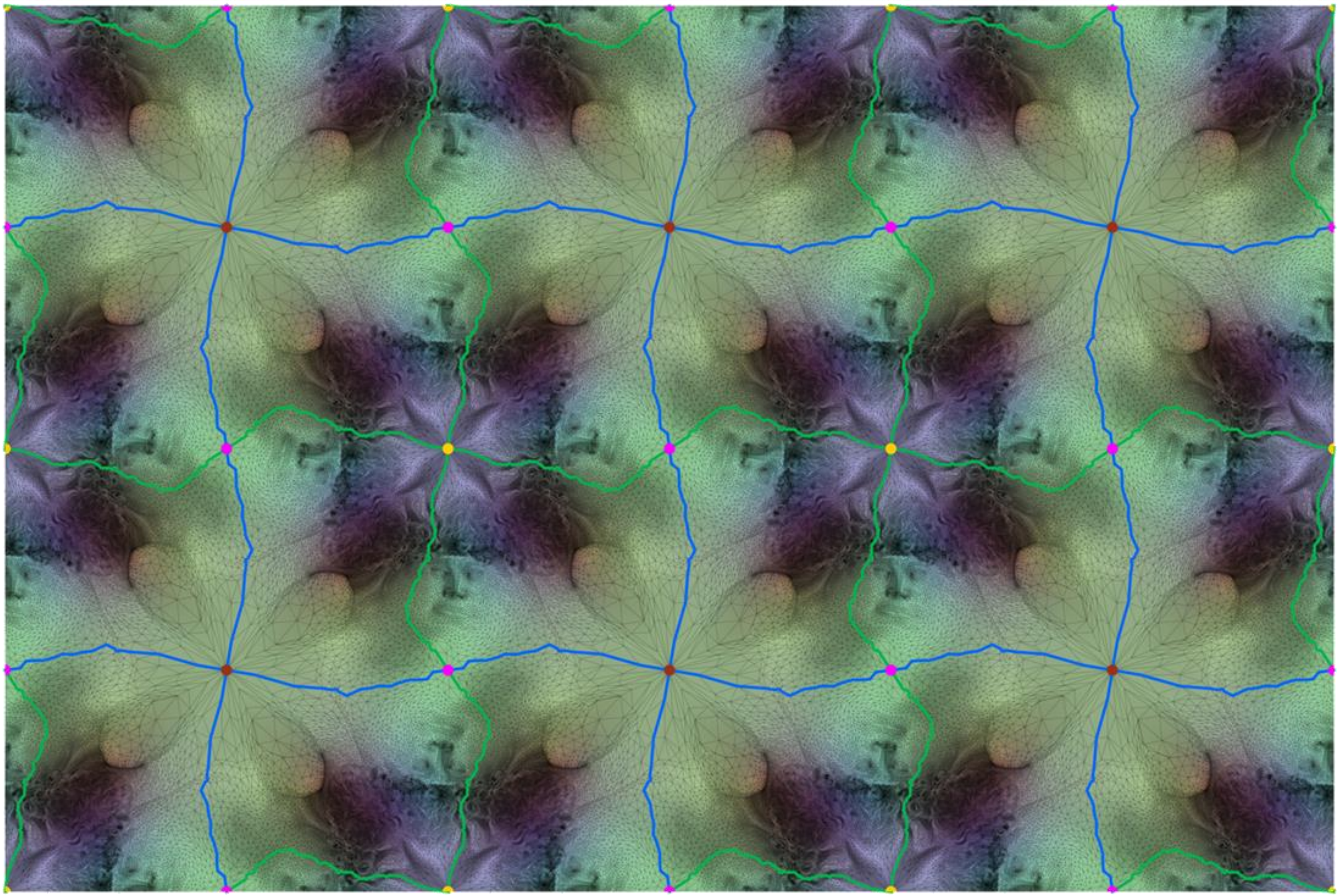
There exists a unique solution, and it is **injective**!

Solve sparse linear system!

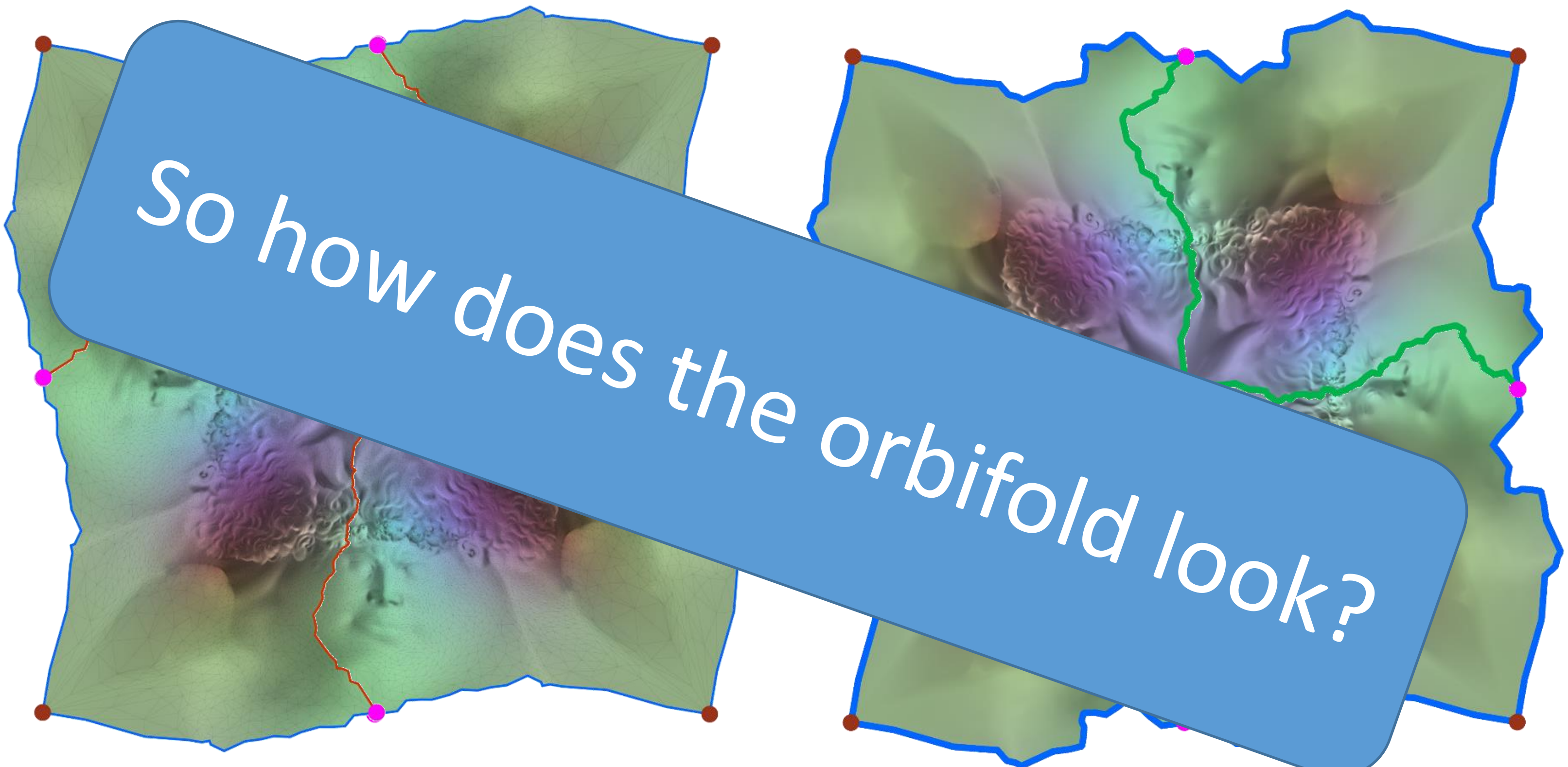


# Why does it work?

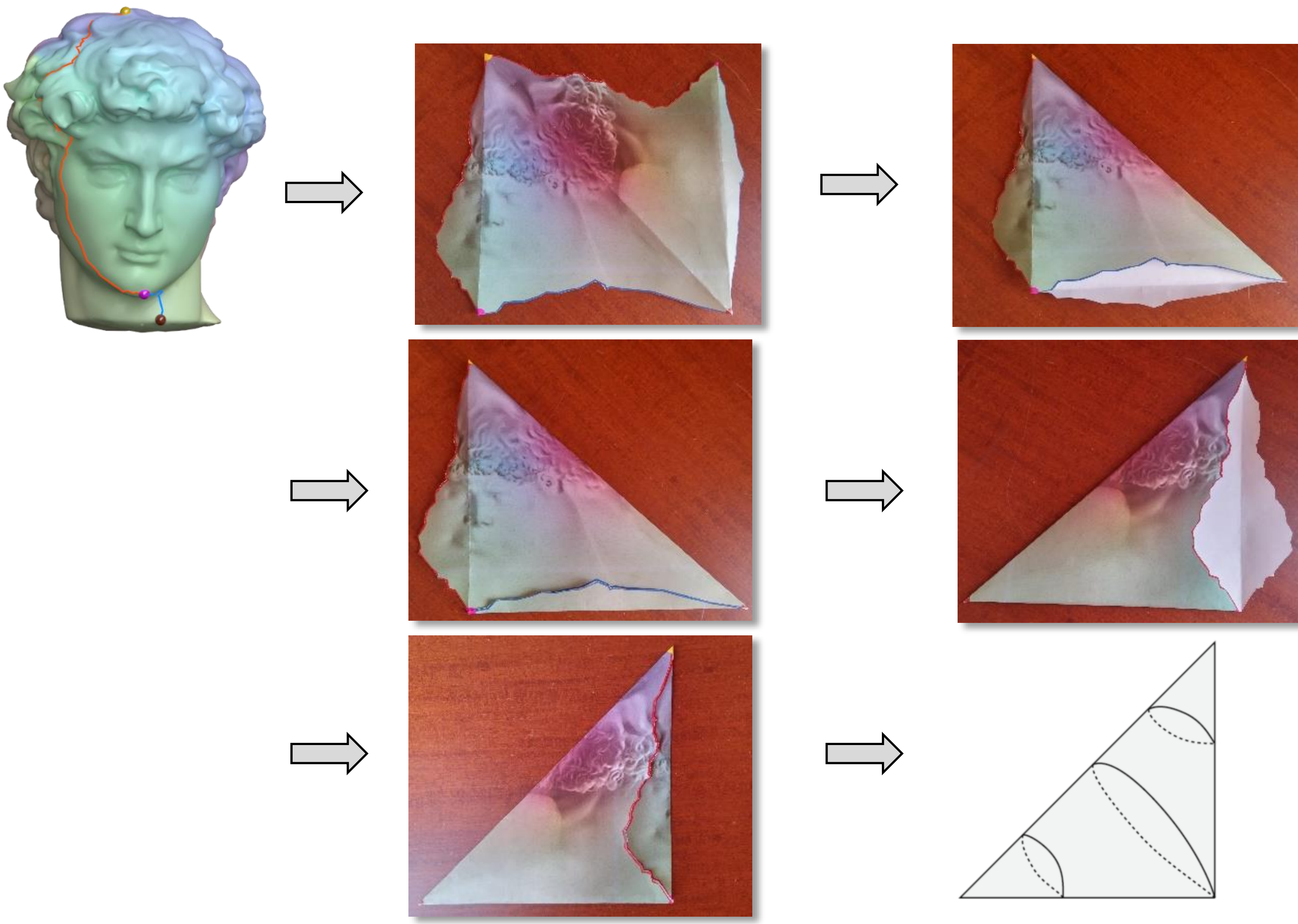
## Euclidean orbifold = cone manifold which tiles $\mathbb{R}^2$



# Different cuts yield same embedding



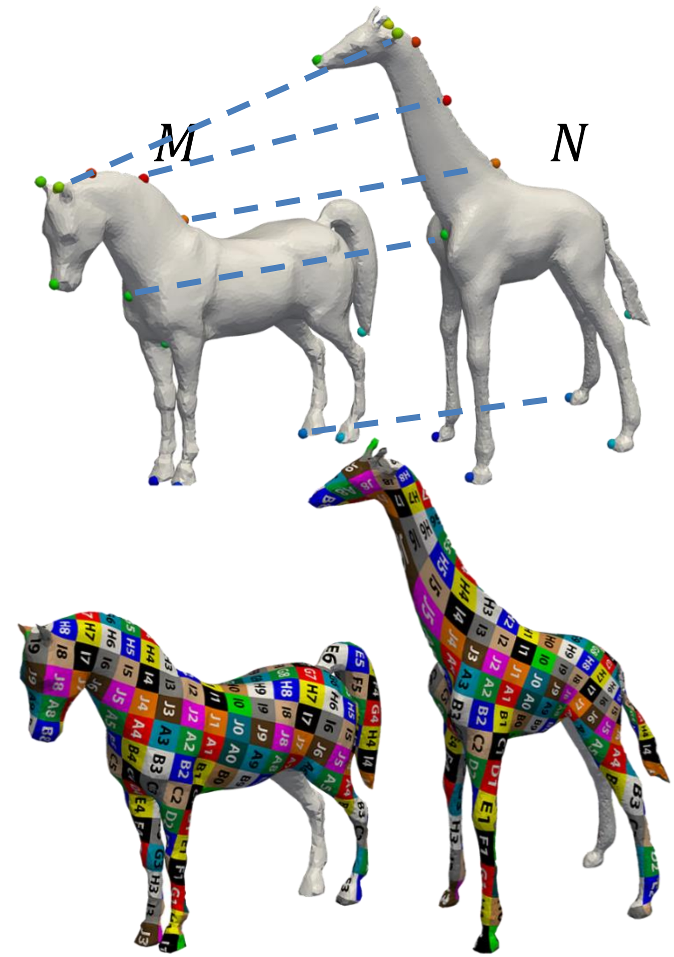
# Embed seamlessly into a “pillow”



# Surface maps

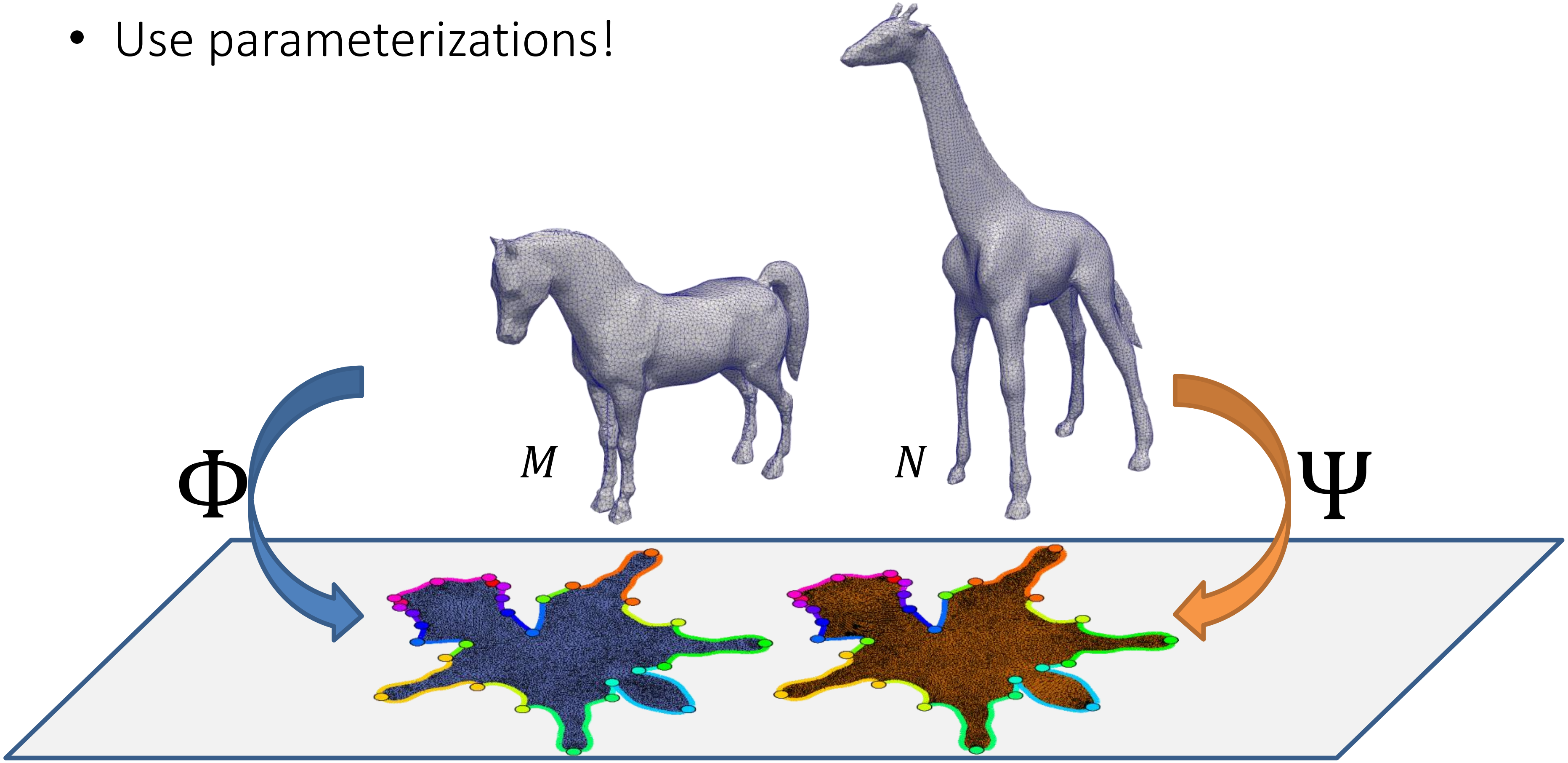
# Surface maps

- Input:
  - Two surface meshes $M, N$
  - Coarse set of corresponding landmarks
- Output: a map $f: M \rightarrow N$
  - Bijective (1-1 and onto)
  - High quality (low isometric distortion)
  - Maps landmarks correctly

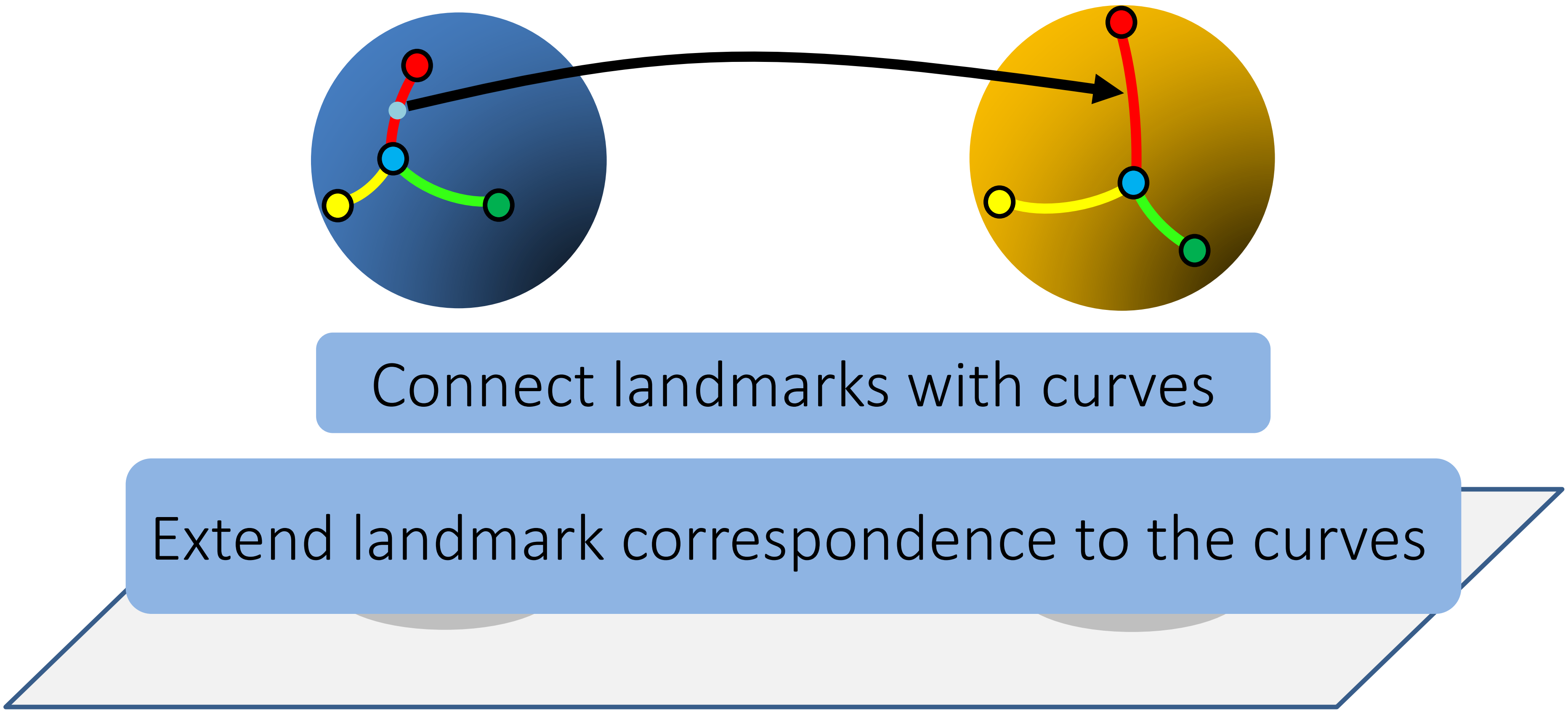


# How to represent a surface map?

- Use parameterizations!



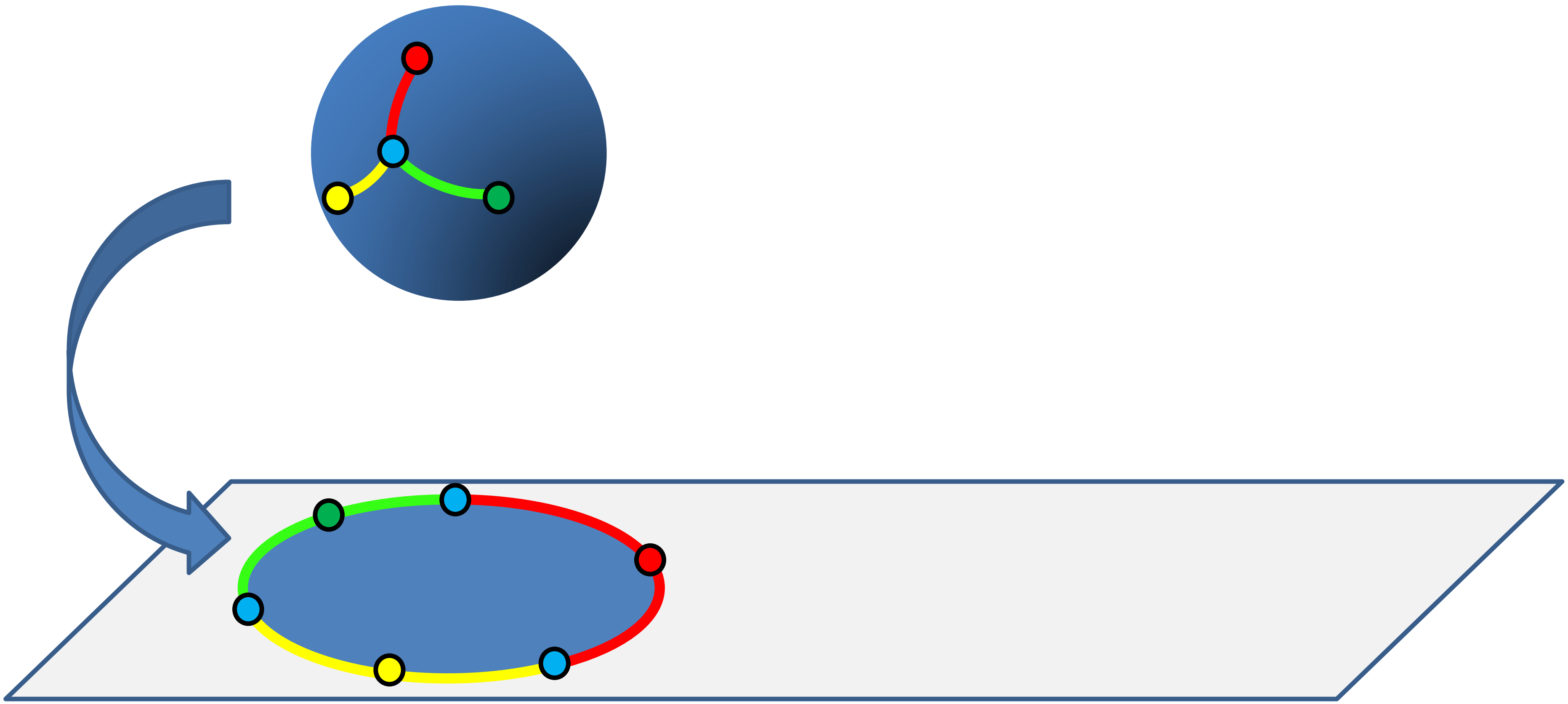
# Use flattenings to $\mathbb{R}^2$



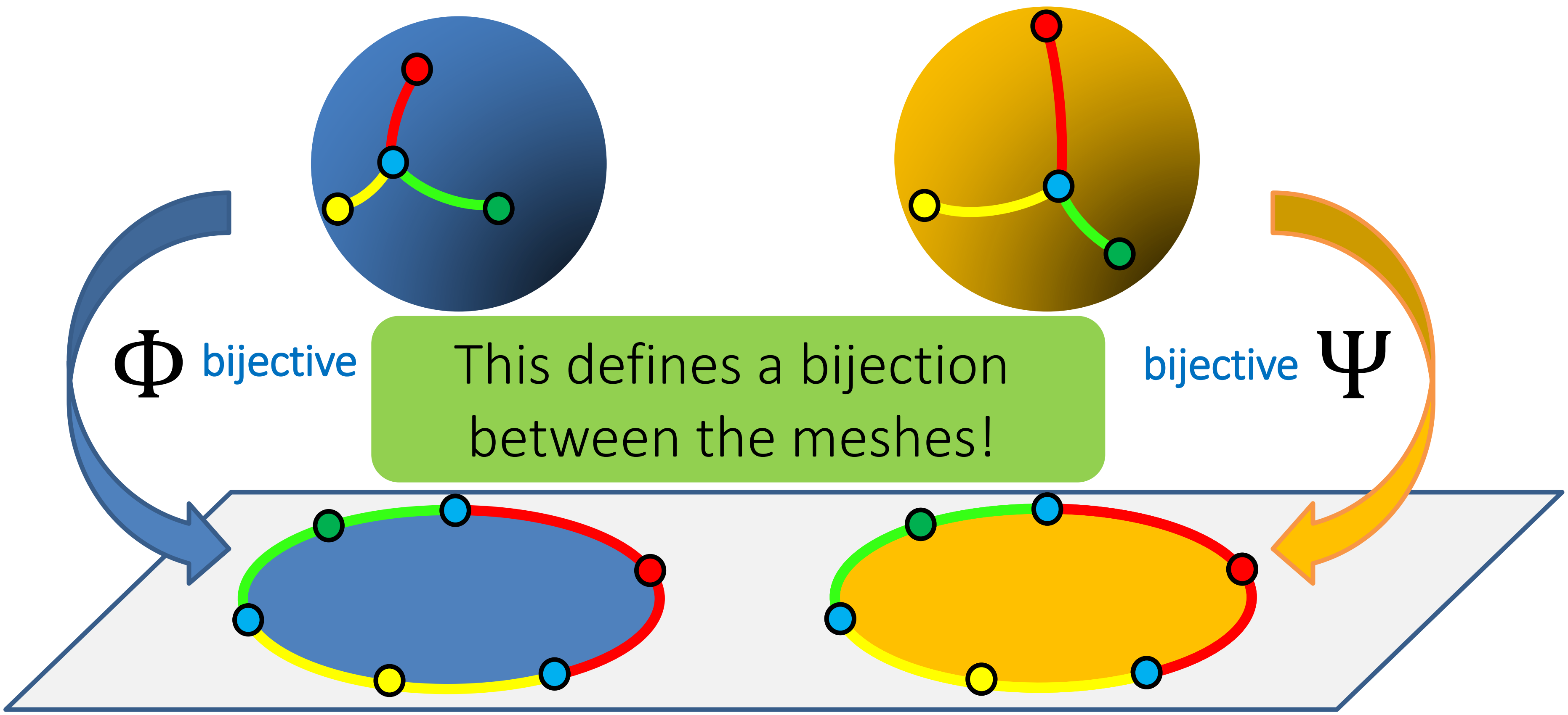
Connect landmarks with curves

Extend landmark correspondence to the curves

# Cut the mesh and map to disk

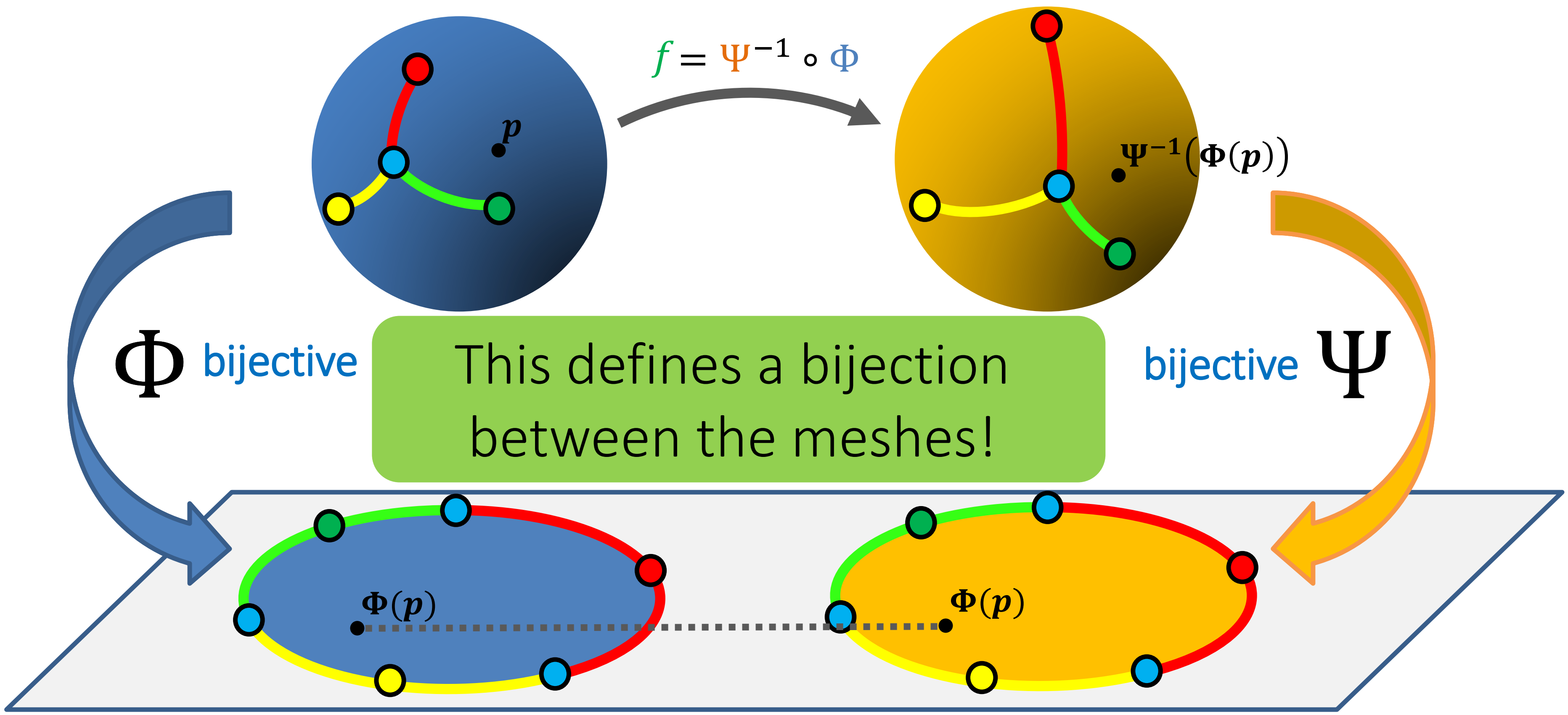


# Flattenings to $\mathbb{R}^2$




This defines a bijection between the meshes!

# Recovering the bijection

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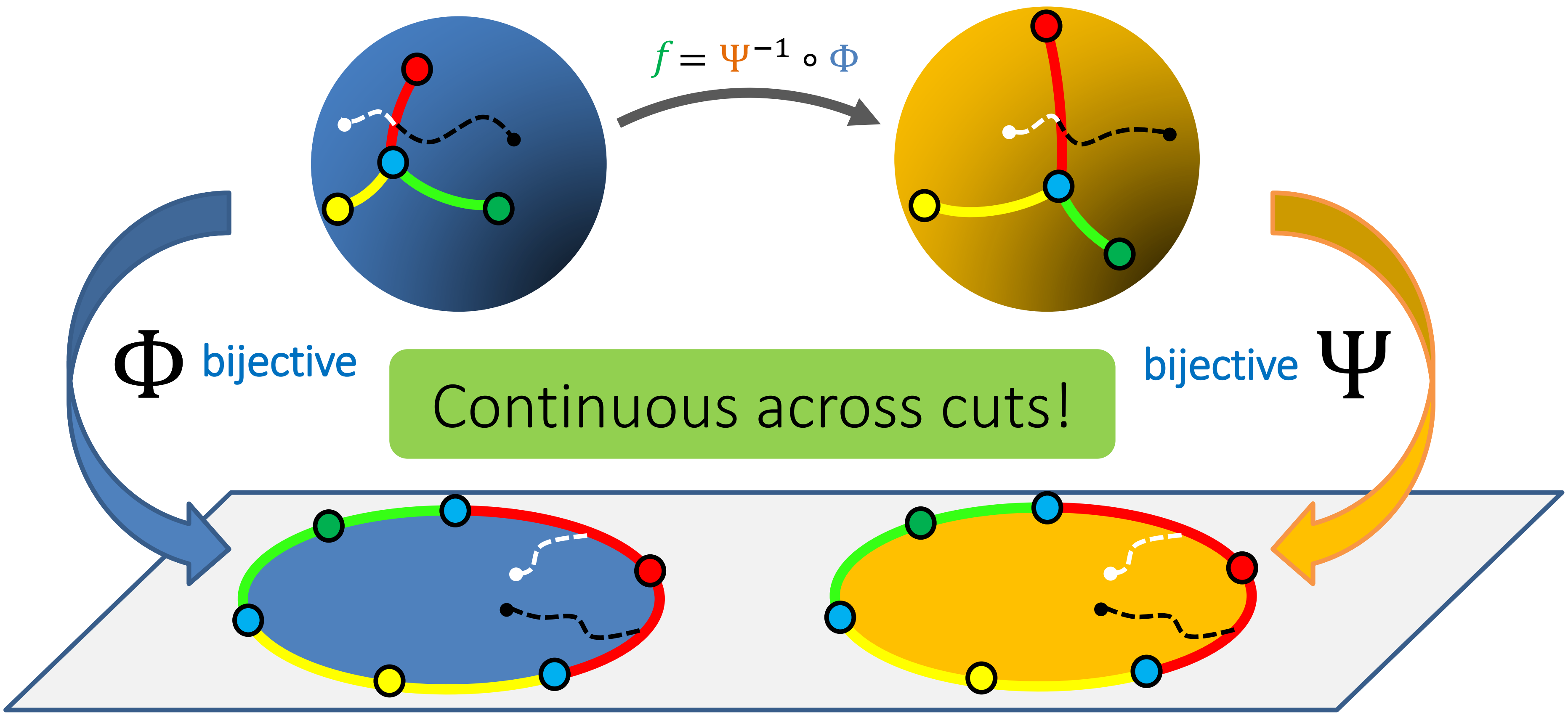
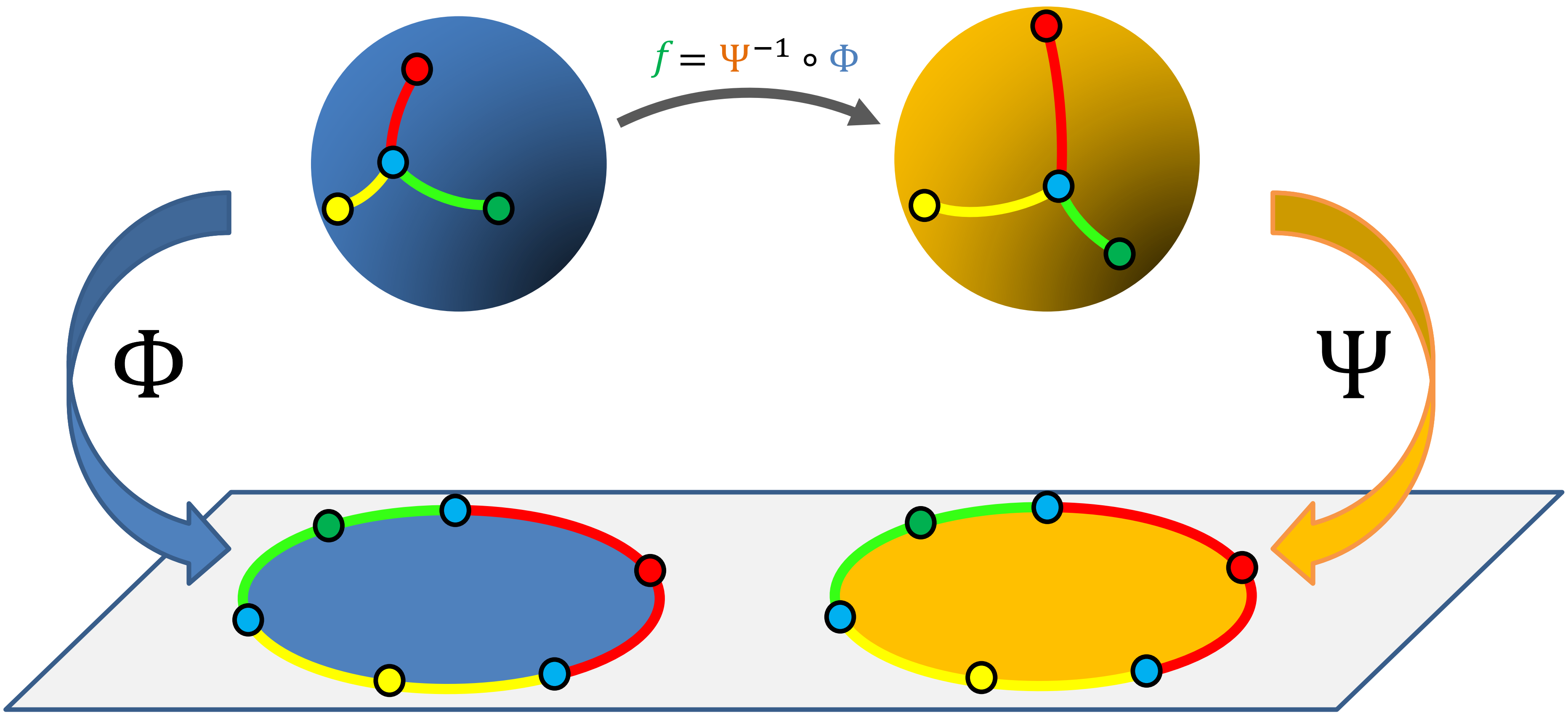
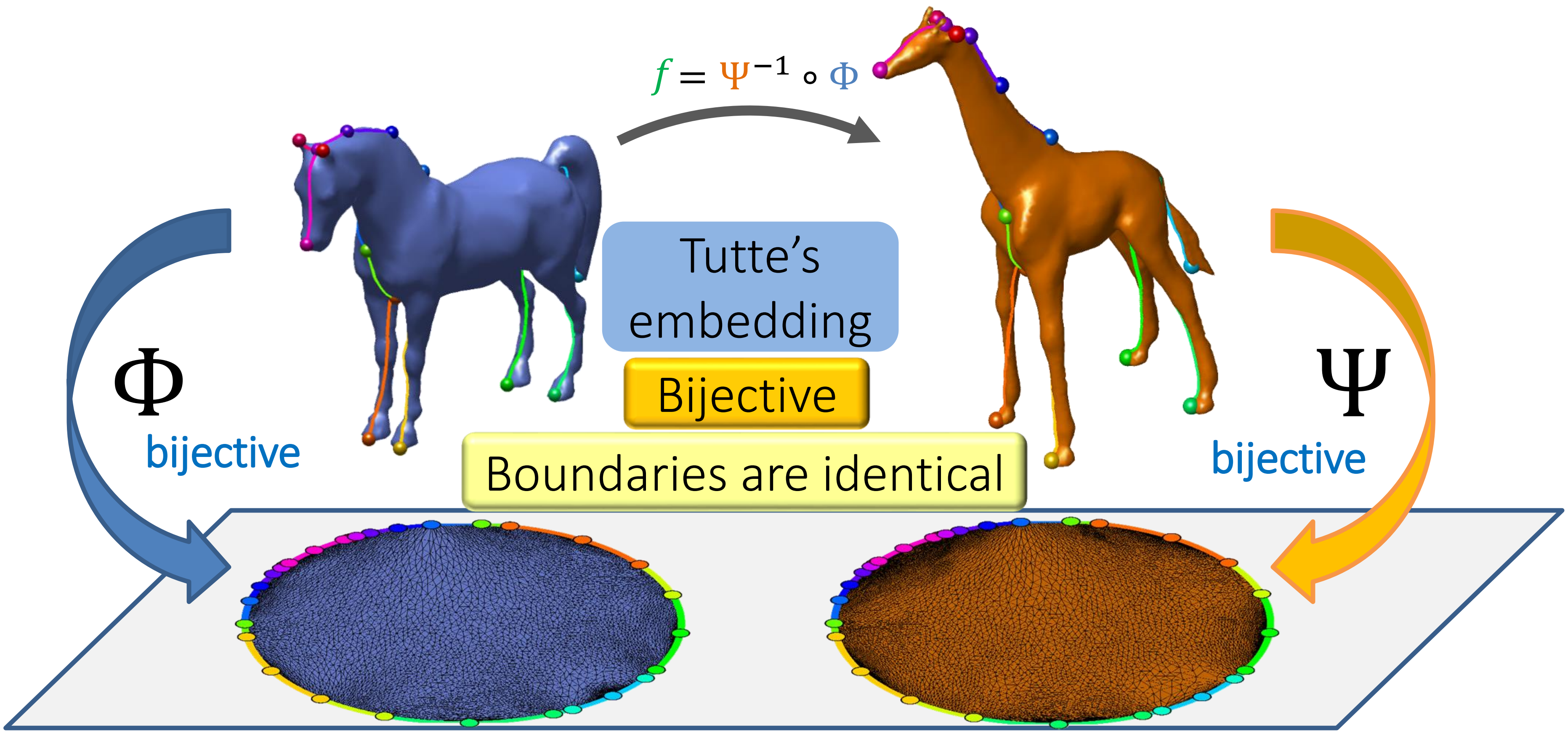
# Recovering the bijection



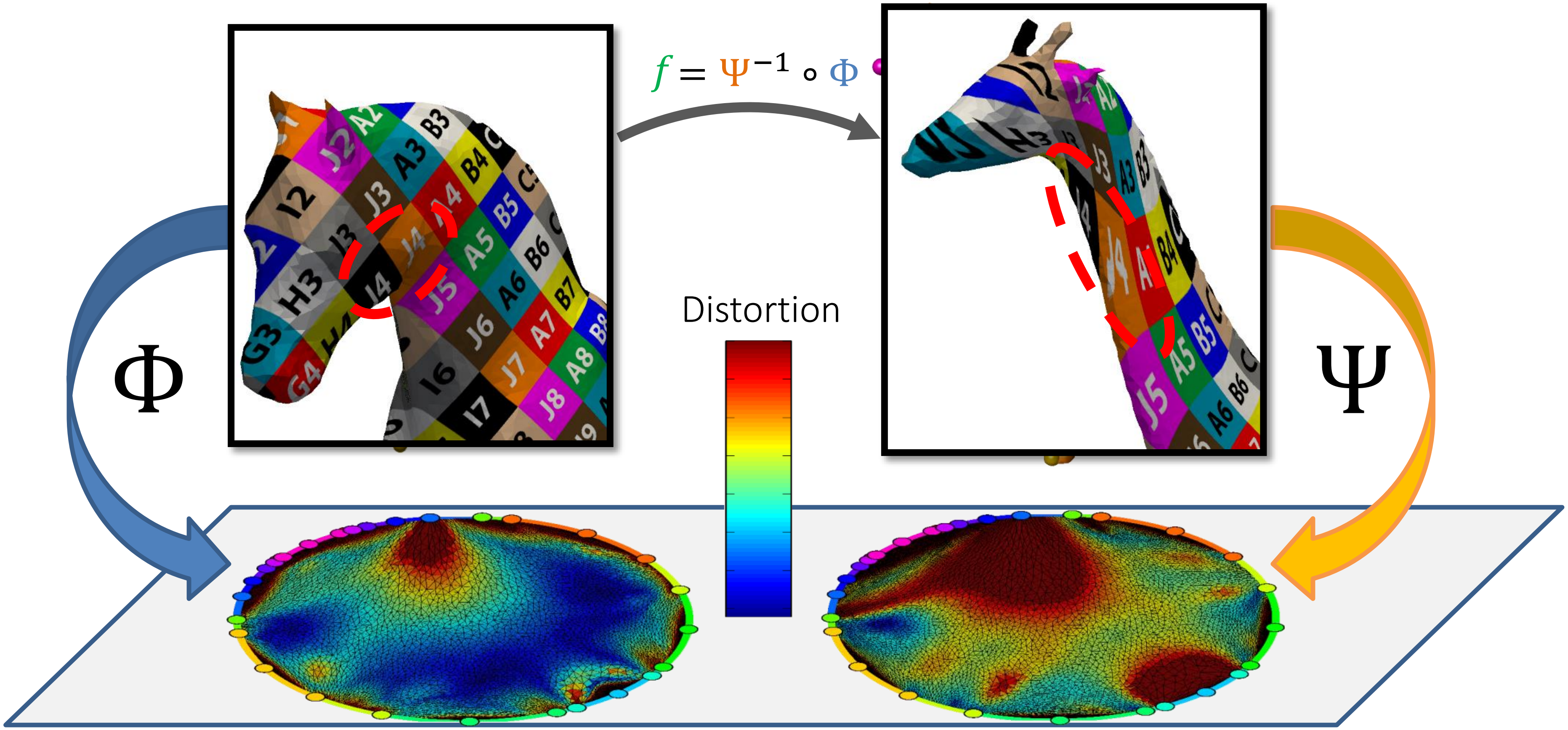
# Is this good enough?



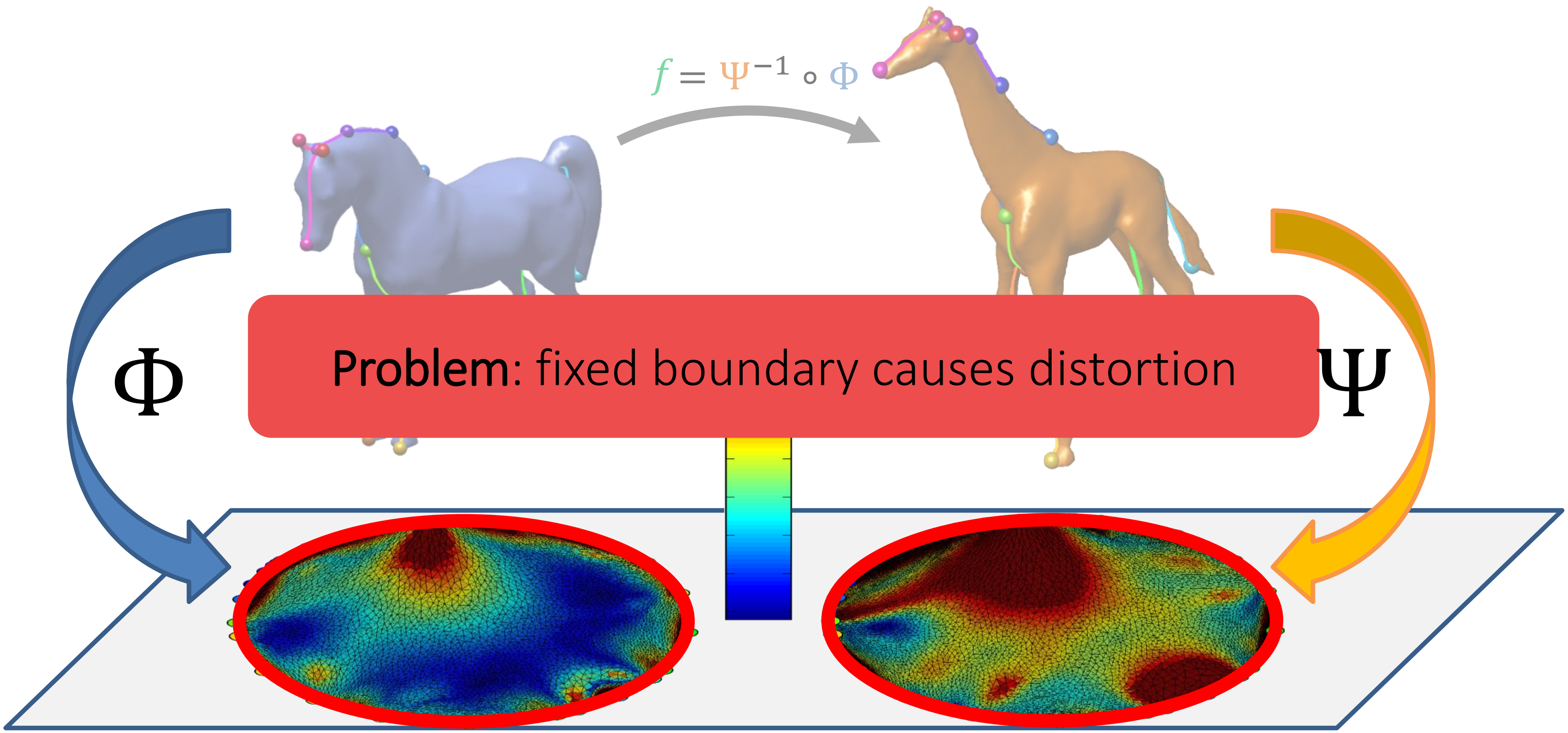
# Is this good enough?



# Is this good enough?

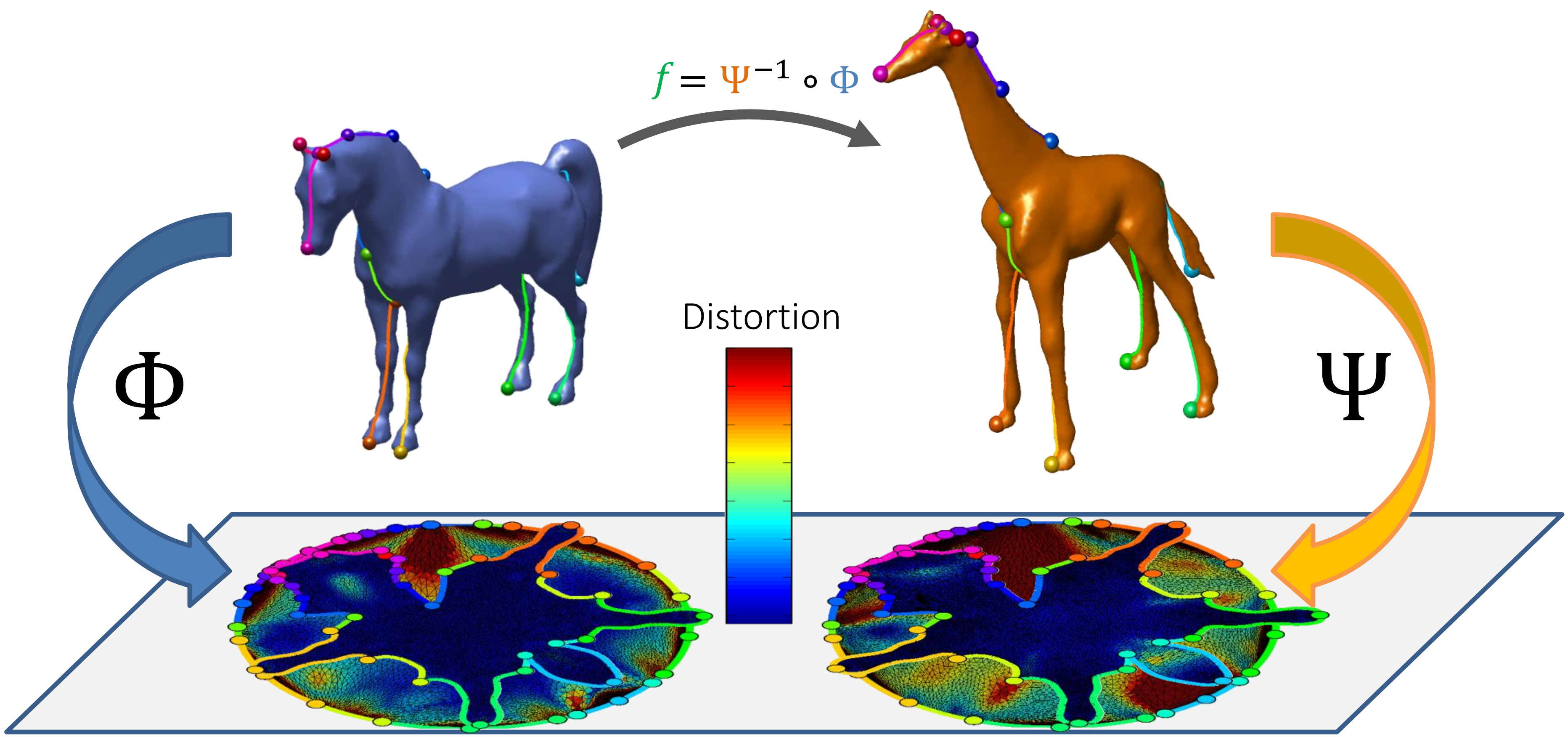


# Reduce the flattenings' distortion!

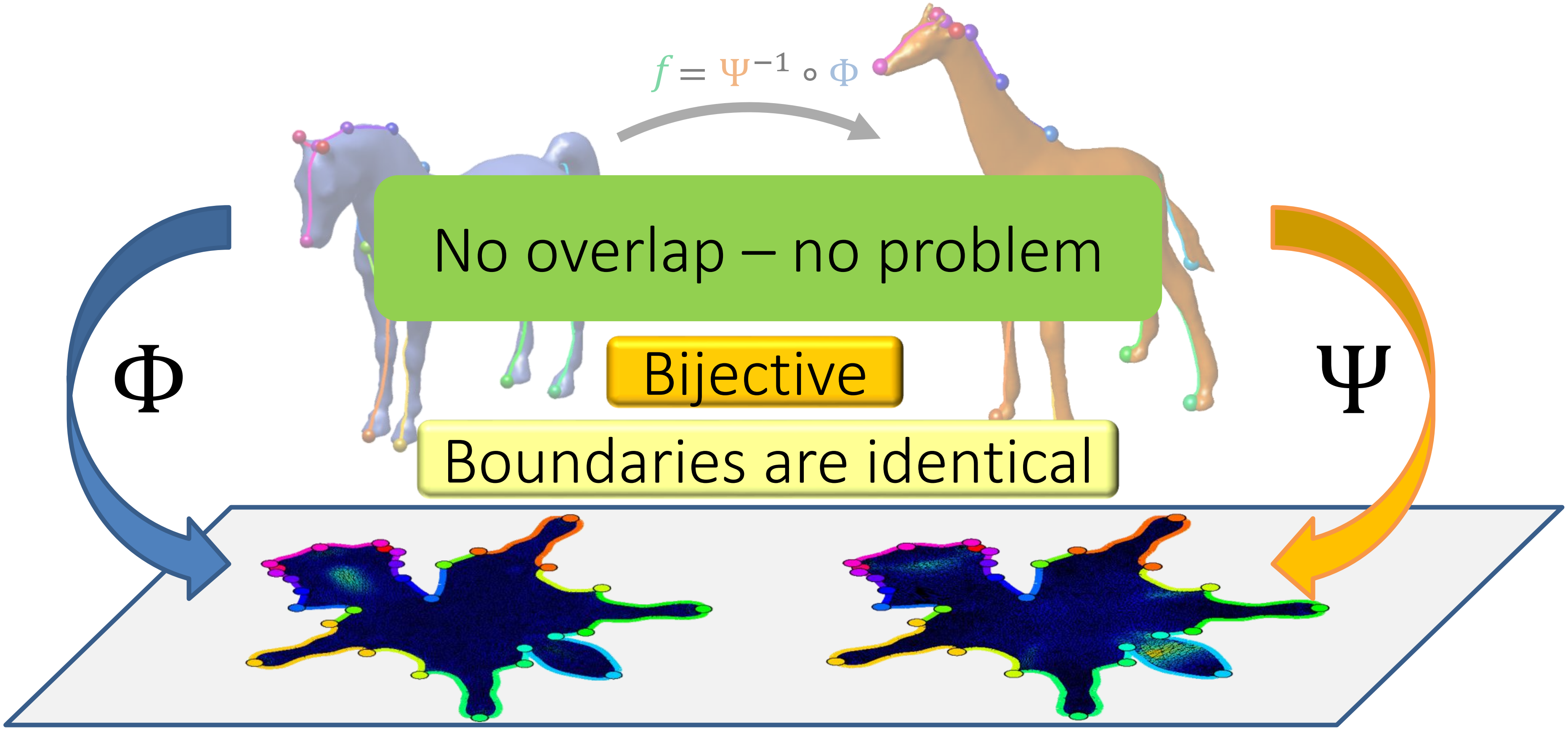



**Problem**: fixed boundary causes distortion

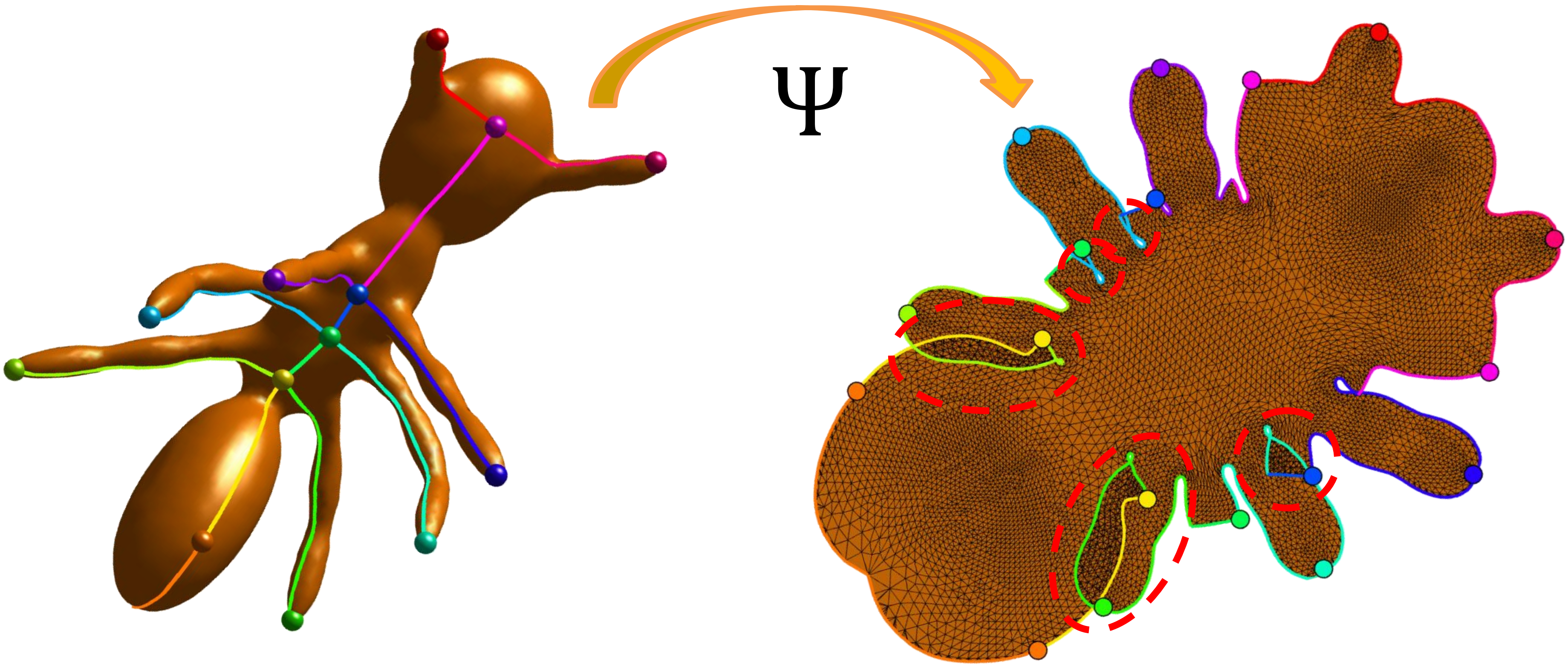
# Let the boundaries move!



# Let the boundaries move

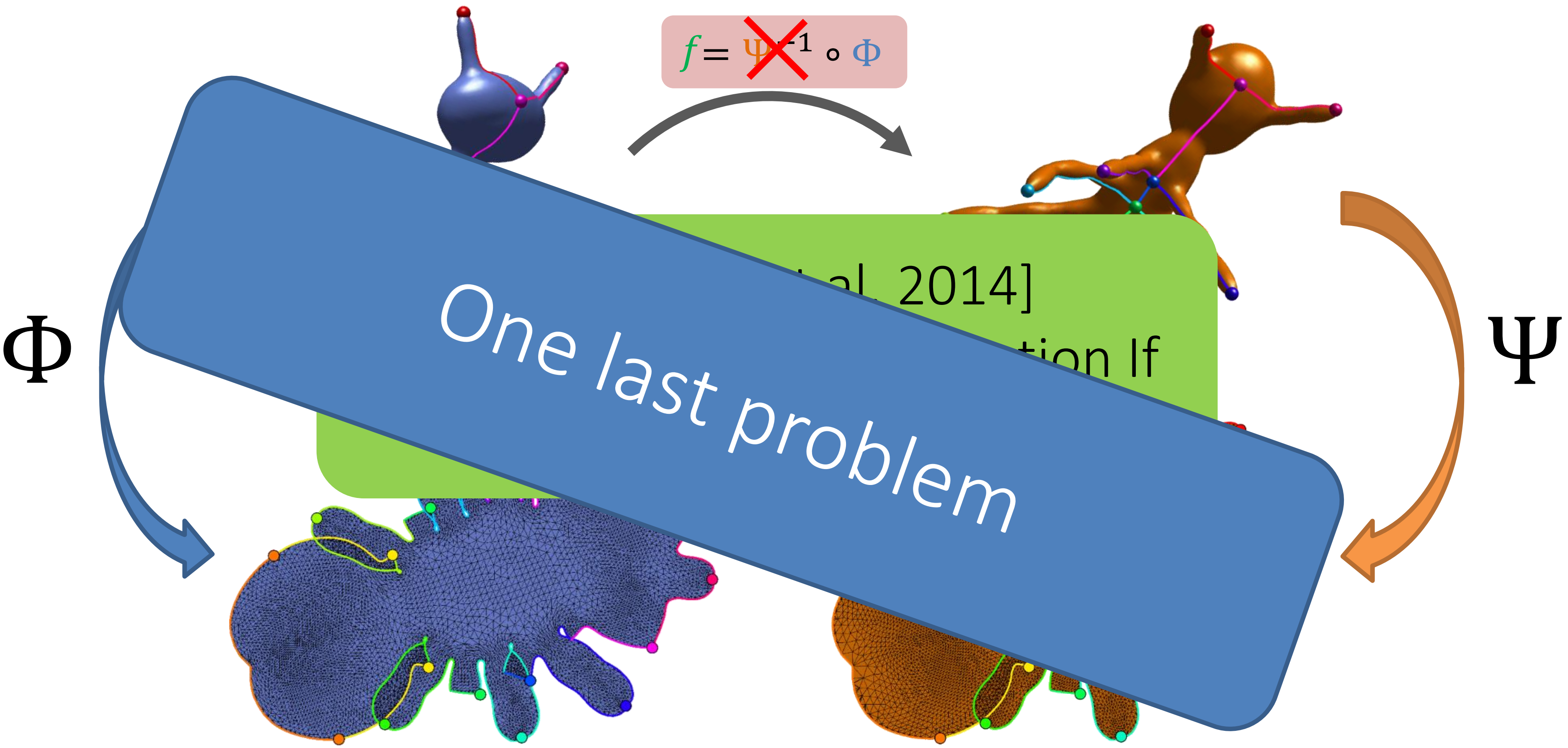


# Overlaps

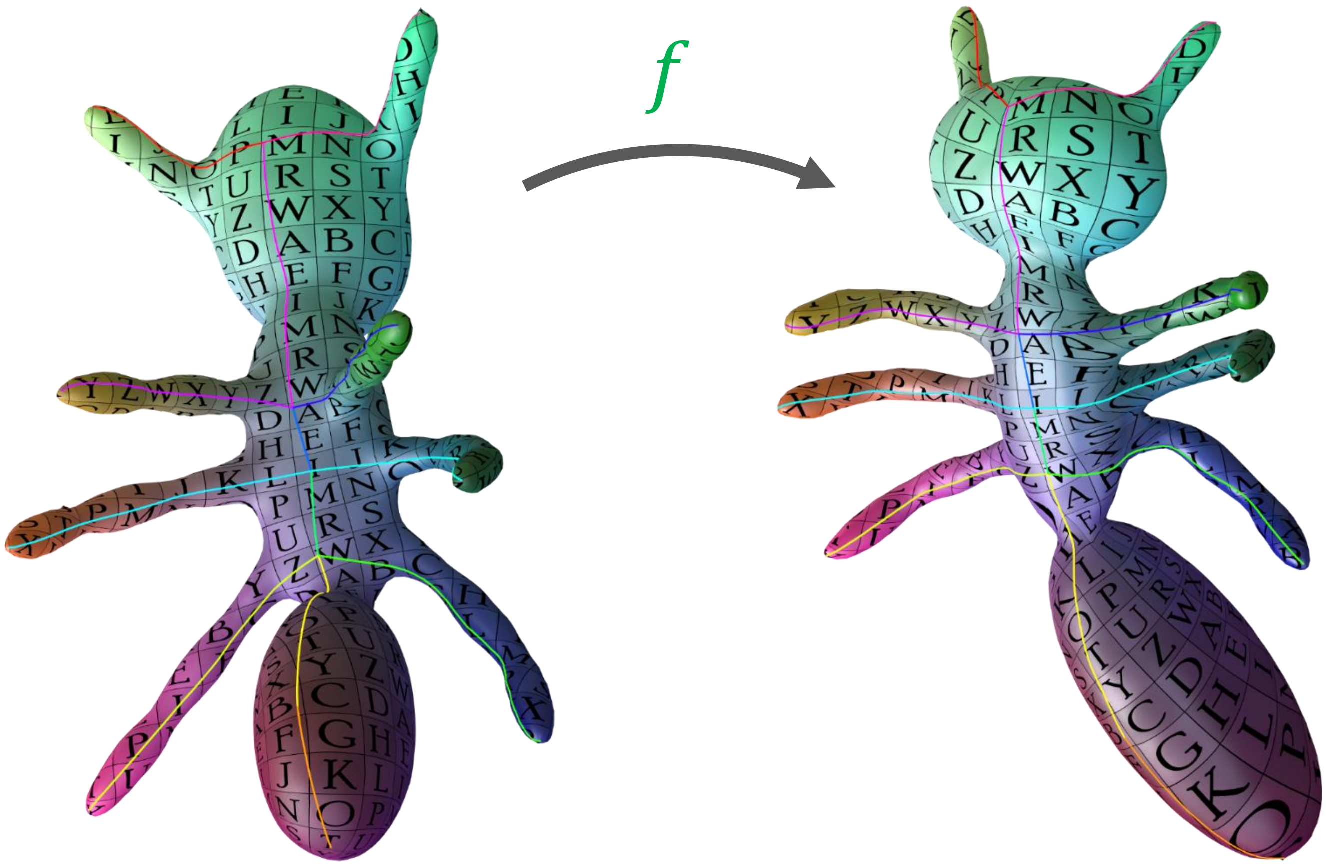


# The non-injective case

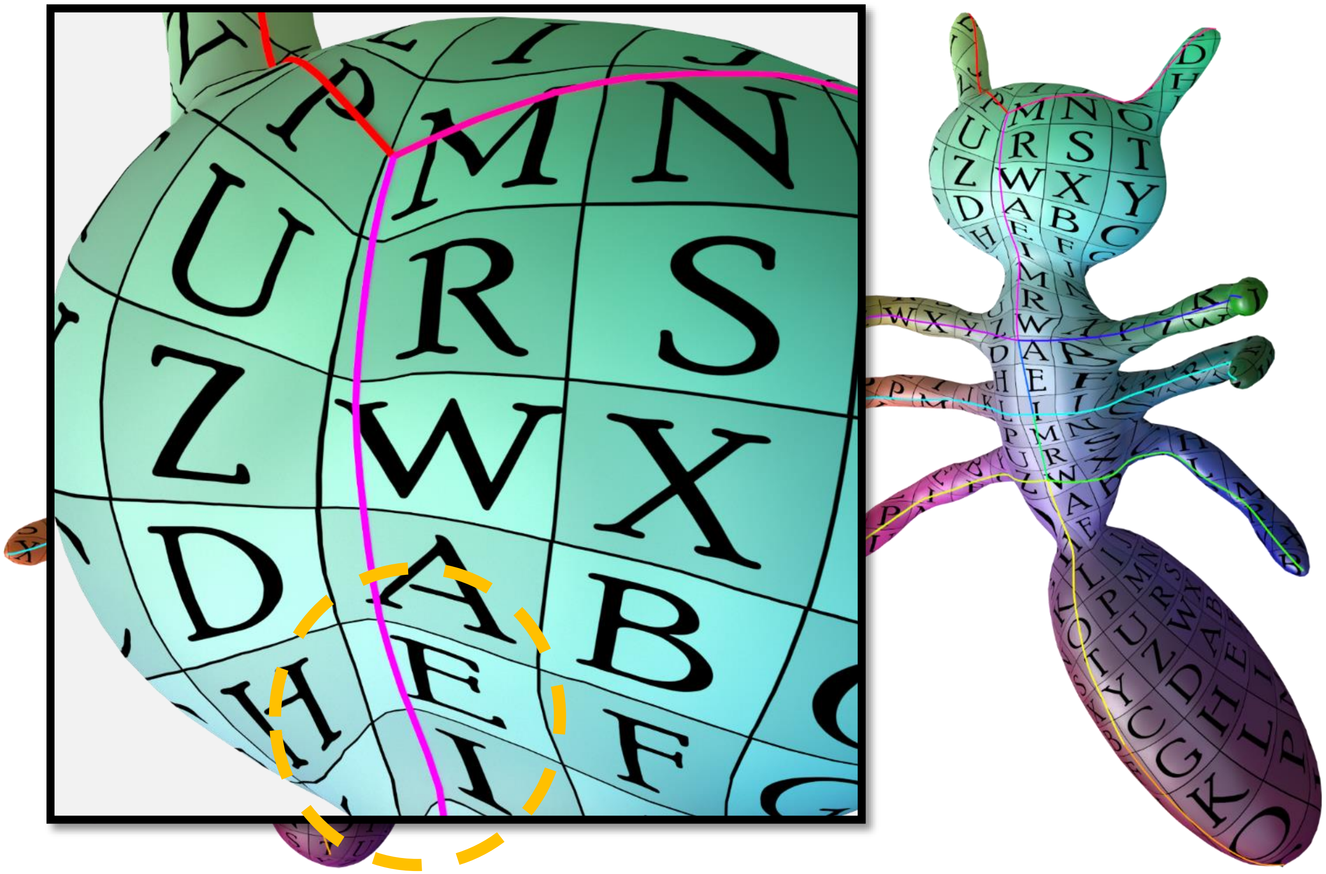
$f = \cancel{\Psi^{-1}} \circ \Phi$



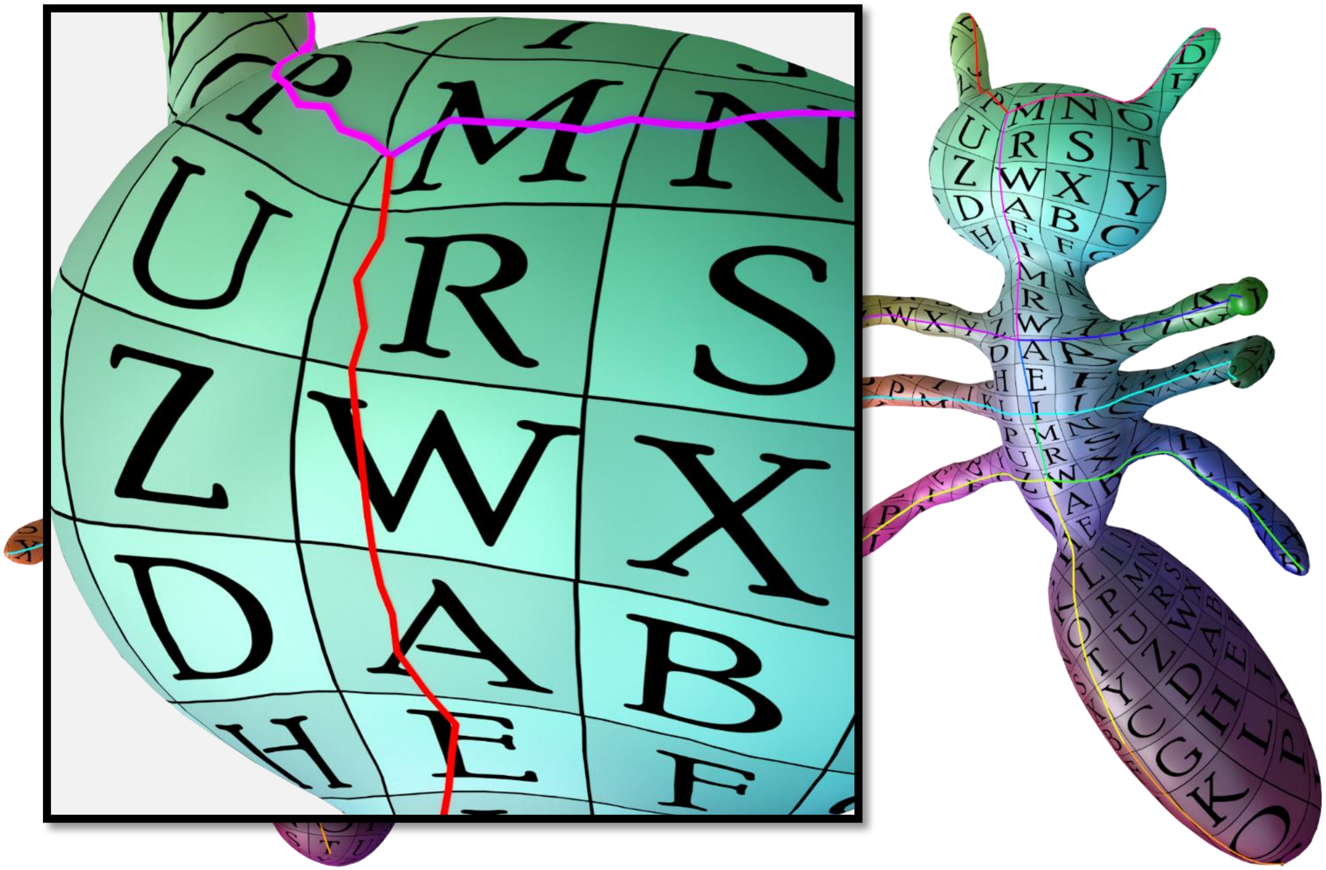
# Cuts affect mapping!



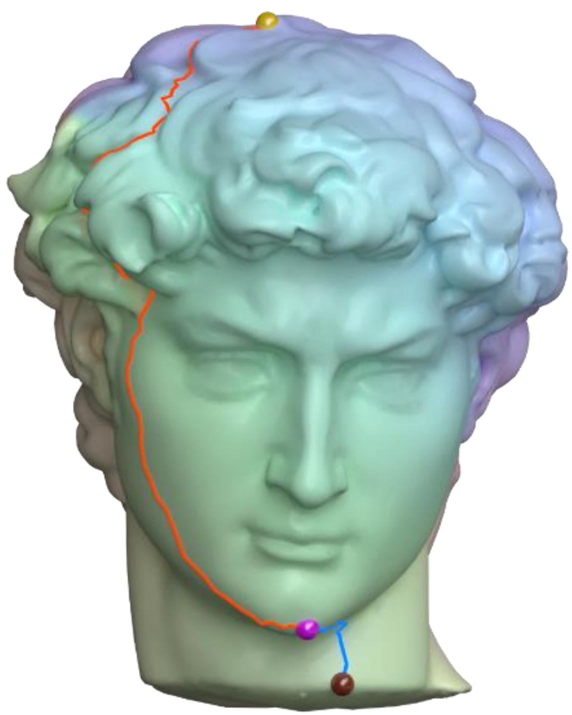
# Cuts affect mapping!

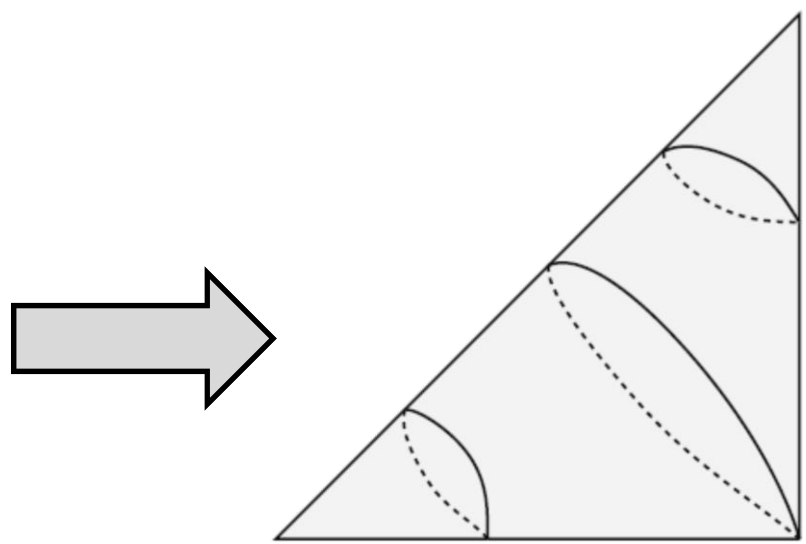


# How to achieve **seamless** result?

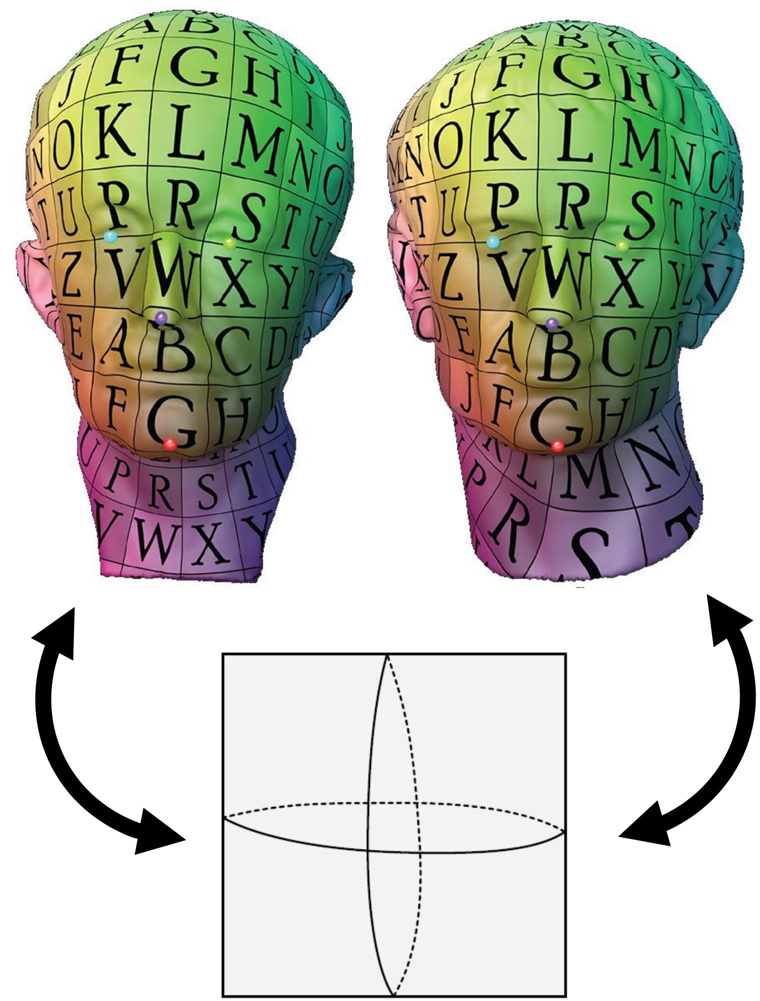


# Orbifolds are seamless



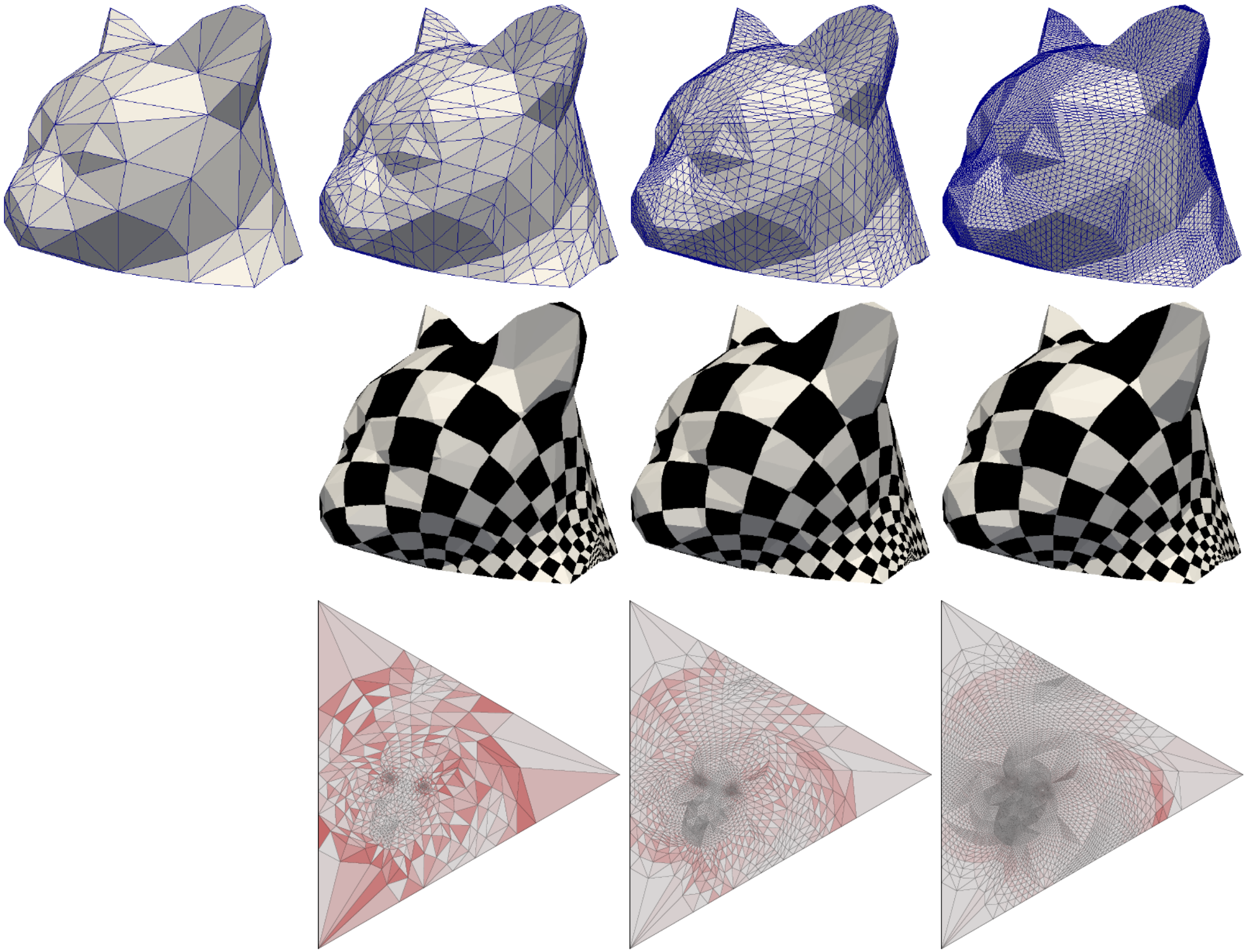


# Define map via orbifold embeddings



# Summary

# Piecewise linear is simple yet powerful

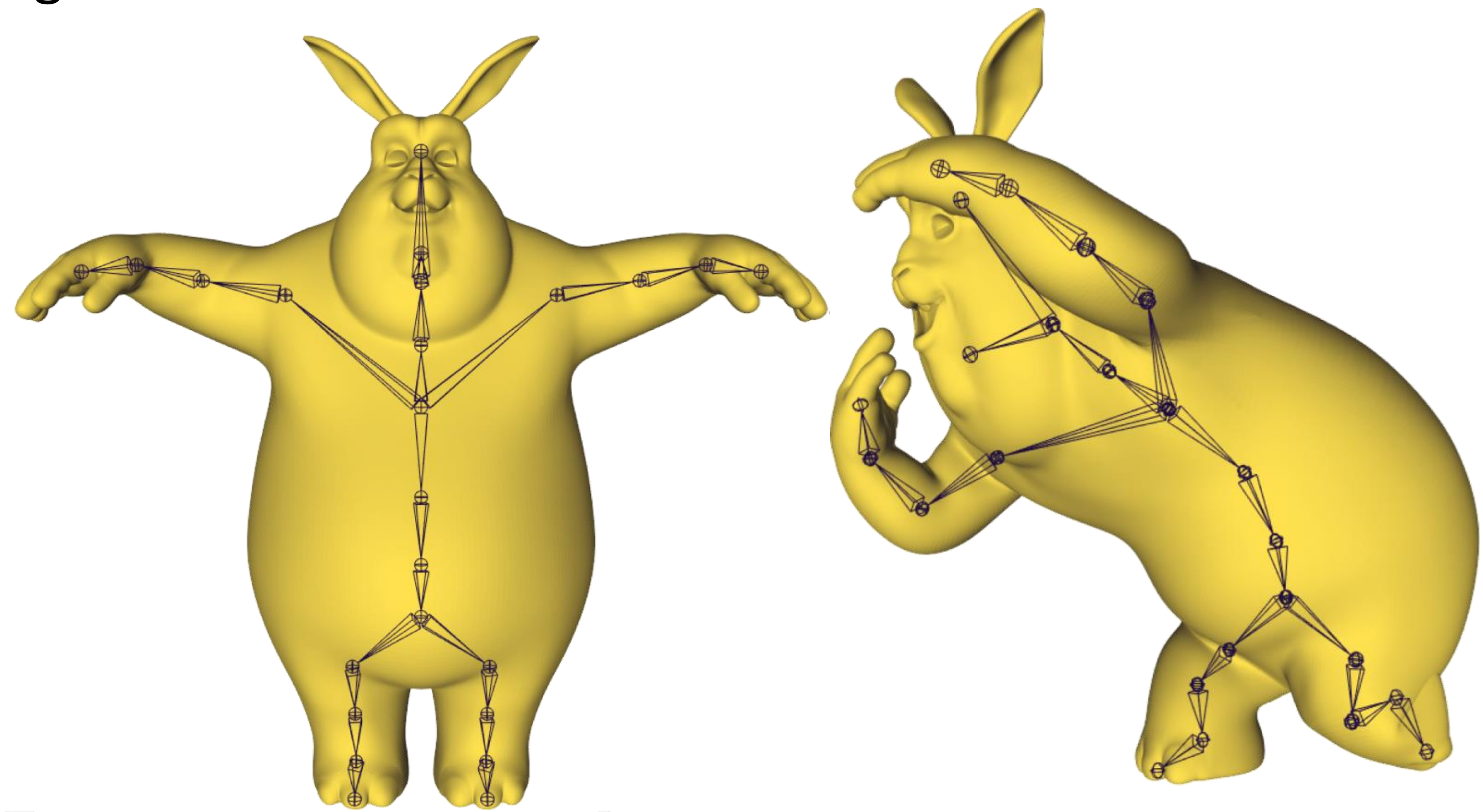


# Beyond piecewise linear

- Meshless (e.g., thin plate splines)

# Beyond piecewise linear

- Higher order FEM

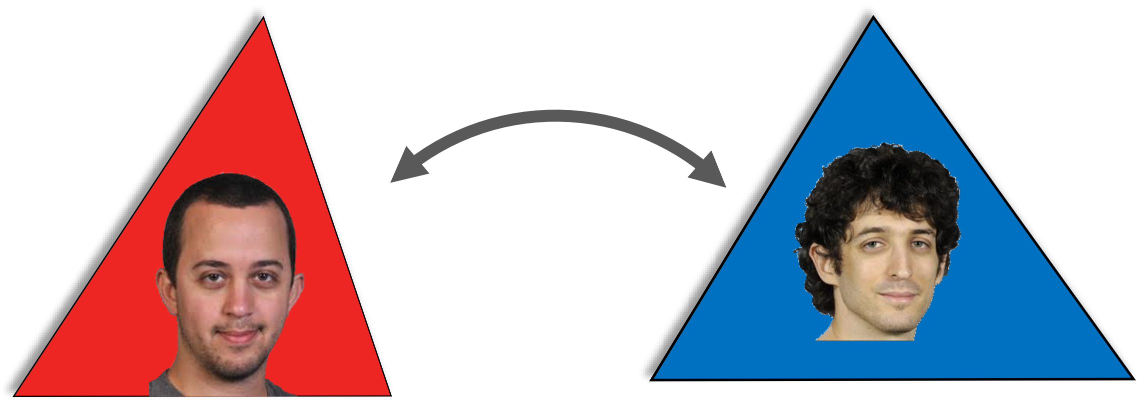


[Liu et al. 2014]

# What’s next?

- Faster optimization
- Coping with non-Euclidean domains
- New distortion metrics

# THE END



(Some things cannot be mapped)